DECAY AND ASYMPTOTIC BEHAVIOR OF A SOLUTION OF THE KELLER-SEGEL SYSTEM OF DEGENERATED AND NON-DEGENERATED TYPE

Takayoshi Ogawa

Mathematical Institute, Tohoku University
Sendai 980-8578, Japan

Abstract. We classify the global behavior of the weak solution of the Keller-Segel system of degenerated type. For the stronger degeneracy the weak solution exists globally in time and it shows the time uniform decay under some extra conditions. If the degeneracy is weaker the solution exhibit a finite time blow-up if the data is non-negative. The situation is very similar to the semi-linear case. Some additional discussion is also presented.

1. Keller-Segel system

1.1. **Survey for Non-degenerated Case.** This note is concerning the temporal behavior of a global solution of the degenerated parabolic elliptic system. Before introducing the problem we consider, let us start from the original model of the chemotaxis called as the Keller-Segel system introduced in [16]. The semilinear type of the original Keller-Segel system is the following form: For $\lambda \geq 0$,

$$(1.1)\qquad \begin{cases} \partial_t u - \Delta u + \nabla(u\nabla\psi) = 0, \quad x \in \mathbb{R}^n,\ t > 0, \\ \partial_t \psi - \Delta\psi + \lambda\psi = u, \quad x \in \mathbb{R}^n,\ t > 0, \\ \qquad u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \\ \qquad \psi(0,x) = \psi_0(x), \quad x \in \mathbb{R}^n. \end{cases}$$

Here the unknown function $u(t,x); \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ denotes the density of a mucus amoeba and $\psi(t,x); \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ stands for the potential of chemical substances. In order to exploit the contrast between the existence and non-existence of the solution, Jäger-Luckhaus [14], Wolansky [38] and Nagai [22] considered the parabolic-elliptic version of the above system:

$$(1.2)\qquad \begin{cases} \partial_t u - \Delta u + \nabla(u\nabla\psi) = 0, \quad x \in \mathbb{R}^n,\ t > 0, \\ -\Delta\psi + \lambda\psi = u, \quad x \in \mathbb{R}^n,\ t > 0, \\ \qquad u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases}$$

It has been studied in detail for the asymptotic behavior of the solutions for the above systems ([14], [40], [1], [23], [9]). In fact this system (1.2) has a strong connection with the self-interacting particles that studies largely by Biller [1], [2] and reference therein.

The above systems are also connected to a simplest model equation of the semiconductor devise simulation of bipolar type (cf. [21], [15]):

$$
\begin{cases}
\partial_t n - \Delta n - \nabla(n\nabla\psi) = 0, \quad x \in \mathbb{R}^n,\ t > 0, \\
\partial_t p - \Delta p + \nabla(p\nabla\psi) = 0, \quad x \in \mathbb{R}^n,\ t > 0, \\
-\Delta\psi = \varepsilon(p - n) + g, \quad x \in \mathbb{R}^n,\ t > 0, \\
\qquad n(0,x) = n_0(x), \quad p(0,x) = p_0(x), \quad x \in \mathbb{R}^n,
\end{cases}
\tag{1.3}
$$

where $n(t,x)$ and $p(t,x)$ denote the density of the negative and positive charge, respectively and $g(x)$ denotes the background charge density which is a given function. When the background charge can be neglected, the equation is considered as the two species version of the Keller-Segel model except the sign of the nonlinear interaction. The semi-conductor devise model chooses a stabler sign of the nonlinearity that makes the system admits large data global solutions. Note that the unstable case, there is an analogous blow up result holds for the above two species system (see Kurokiba-Ogawa [20] and Kurokiba-Nagai-Ogawa [19]). In the both cases (1.2) and (1.3), the critical case for the equation is $n = 2$ in the scaling point of view. This is corresponding to the well known Fujita exponent $1 + 2/n$ for the semilinear heat equation ([12]) and the two dimensional case the quadratic nonlinearity is exactly corresponding to the critical situation. The existence, the uniqueness and the regularity theory for the corresponding problem in a bounded domain has already been done by many authors. Here we concentrate the Cauchy problem in $\mathbb{R}^2$ to examine the scaling invariance point of view.

The result for the global existence for the Keller-Segel system (1.1) can be summarized as follows:

Theorem 1.1. ([25]) *Let $\lambda > 0$ be constants and $n = 2$. Suppose $(u_0, v_0) \in (L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)) \times H^1(\mathbb{R}^2)$ are positive. Then under the condition either for* (1.1),

$$
\int_{\mathbb{R}^2} u_0(x)dx < 4\pi \tag{1.4}
$$

or for (1.2)

$$
\int_{\mathbb{R}^2} u_0(x)dx < 8\pi, \tag{1.5}
$$

then the positive solution to (1.1) (*or* (1.2)) *exists globally in time. Namely $(u, v) \in C([0,\infty); (L^2 \cap L^1) \times (H^1 \cap L^1)) \cap C^1((0,\infty); H^2 \times H^2)$ and it satisfies that for all $T > 0$, there exists a finite constant $C = C(T)$ such that*

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \{(1 + u(t)) \log(1 + u(t))\} + \frac{1}{2}\|\nabla v(t)\|_2^2 + \frac{1}{2}\lambda\|v(t)\|_2^2 \le C(T), \\
&\quad t \in [0,T].
\end{aligned}
\tag{1.6}
$$

In the both cases, the role of the generalized free energy (1.6) is important to obtain the time apriori estimate for the solutions. Note that it has already proved that if the initial data satisfies

$$
\int_{\mathbb{R}^2} u_0(x)dx > 8\pi
$$

then the positive solution blows up in a finite time (cf. Biler [1], Nagai [22] and Nagai-Senba-Yoshida [27]).

On the other hand, to discuss the analogous result for the simpler system $\lambda = 0$ of (1.2) we encounter a different kind of technical difficulty. For this case, it is also known that the solution with $u_0 \ge 0$ blows up in a finite time if $\int_{\mathbb{R}^2} u_0(x)dx > 8\pi$ (Biler [1], Nagai [22], [24] and Nagai-Senba-Yoshida [26]). For the whole space case, the restriction that the solution having the finite second moment $\int_{\mathbb{R}^2} |x|^2 u(t)dx < \infty$ is removed by the scaling method in Kurokiba-Ogawa [20].

Besides when the domain is bounded in $\mathbb{R}^2$ with the Neumann boundary condition, Senba-Suzuki [31] showed that the L^1 density shows a concentration with the measure $8\pi\delta_0$ if the data is the radially symmetric. This can be generalized for the non-radial case by Senba-Suzuki [32].

The second system (1.2) with $\lambda = 0$ also has analogous property of its structure. However the proof of the global existence is rather complicated since the behavior of the solution of the second equation is different from the first one. Namely we can not use the free energy functional directly to derive any a priori bound for the solution which is not considered in the literatures before. We discuss on this direction in [25] in details. One may summarize those existence and non-existence result for the whole space case as follows:

Theorem 1.2. ([25], [20]) *Let $\lambda = 0$ in (1.2). Suppose $u_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ is non-negative everywhere.*
(1) Then under the condition

$$\int_{\mathbb{R}^2} u_0(x)dx < 8\pi, \tag{1.7}$$

the positive solution to (1.2) exists globally in time. Namely $(u, \psi) \in C([0,\infty); (L^2 \cap L^1) \times \dot{W}^{1,\infty}) \cap C^1((0,\infty); H^2 \times \dot{W}^{2,1})$ and it satisfies that for all $T > 0$ there exists a finite constant $C = C(T)$ such that

$$\int_{\mathbb{R}^2} \Big\{(1+u(t))\log(1+u(t)) - u(t)\Big\}dx \le C(T), \quad t \in [0,T].$$

(2) On the other hand, if the positive initial data satisfies

$$\int_{\mathbb{R}^2} u_0(x)dx > 8\pi, \tag{1.8}$$

then the solution does not exists globally. Namely it blows up in a finite time.

The threshold case $\|u_0\|_1 = 8\pi$ is considered recently by Biler-Karch-Laurençot-Nadzieja [5] for the radially symmetric case.

1.2. **Degenerated Case.** The second problem we would like consider here is the degenerated version of the modified model of the Keller-Segel system.

$$\begin{cases} \partial_t u - \Delta u^\alpha + \nabla(u\nabla\psi) = 0, & x \in \mathbb{R}^n, \ t > 0, \\ -\Delta\psi + \lambda\psi = u, & x \in \mathbb{R}^n, \ t > 0, \\ \quad u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.9}$$

where $\alpha \ge 1$ and $\lambda > 0$. An analogous variant of the semiconductor system like (1.3) is also our motivation. In that case, the stabler sign of the nonlinear interaction is chosen.

The striking difference between the semilinear system (1.2) and the degenerated case (1.9) is that the equation essentially includes the hyperbolic structure in it and the finite propagation of the support of the solution may occur. If the solution is strictly positive, the solution is considered similarly as the semi-linear case. As is mentioned for the semilinear case, there exists a finite time blow up solution for a certain initial data and analogously the finite time blow up possibly occurs for the degenerated case. More precisely, when the data is positive and have the large initial value in the sense of L^1, then the solution for the modified version of the Keller-Segel system blows up in a finite time ([22], [1]) when $n = 2$ and for higher dimensional cases, the condition is getting weaker since the system is less stable. For the degenerated case, we expect an analogous situation.

If there is a point where the solution varnish, the equation is essentially degenerated and therefore the notion of weak solution is required.

Definition. Let $\alpha \geq 1$. Given $u_0 \in L^1 \cap L^\alpha(\mathbb{R}^n)$ with $u_0(x) \geq 0$ for $x \in \mathbb{R}^n$, we call $(u(t,x), \psi(t,x))$ as a weak solution of the system (1.9) if there exists $T > 0$ such that

i) $u(t,x) \geq 0$ for any $(t,x) \in [0,T) \times \mathbb{R}^n$,
ii) $u \in C(\mathbb{R}^n \times [0,T))$ with $\nabla u^\alpha \in L^2(\mathbb{R}^n \times [0,T))$,
iii) For arbitrary test function $\phi \in C^{1,1}(\mathbb{R}^n \times [0,T))$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} u(t)\phi(t)dx - \int_{\mathbb{R}^n} u_0\phi(0)dx \\
&\quad = \int_{t_0}^{t_1} \int_{\mathbb{R}^N} (u(\tau)\partial_t\phi(\tau) - \nabla u^\alpha(\tau) \cdot \nabla\phi(\tau) + u(\tau)\nabla\psi(\tau) \cdot \nabla\phi(\tau)) \; dxd\tau
\end{aligned}$$

for $0 \leq t \leq T$, where $\psi = E_n * u$ and $E_n(\cdot)$ is the fundamental solution of $-\Delta + \lambda$ in $\mathbb{R}^n$.

The difference between the degenerated case and semilinear case appears where the solution vanishes. By this regards, it is important to show the finite propagation of the support.

The existence of the weak solution is obtained by an application of the standard theory of the parabolic equation. Note that the equation does not have the comparison principle of solutions for any type nor the semi group representation as is possible for the semilinear case, the proof of the existence requires some approximation procedures involving the parabolic regularity theory. The following result due to Sugiyama [34] is one of the explicit proof of them.

Proposition 1.3 ([34]). *For $\alpha > 1$, there exists $T > 0$ and a weak solution (u, ψ) of the degenerated Keller-Segel system (1.9) for $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$. Moreover*
(1) if $2 \leq \alpha$ the weak solution exists globally in time,

(2) if $1 < \alpha < 2 - \frac{2}{n}$ and the initial data u_0 is sufficiently small in L^1 sense, then the weak solution exists globally.

The exponent $\alpha = 2 - \frac{2}{n}$ is corresponding to the Fujita exponent for the semilinear and quasi-linear parabolic equation of the following type (cf. Fujita [12]):

$$\begin{cases} \partial_t u - \Delta u^\alpha = u^p, & x \in \mathbb{R}^n, \; t > 0, \\ \quad u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.10}$$

where $\alpha \geq 0$. The exponent $p = \alpha + \frac{2}{n} = 1 + \frac{\sigma}{n}$ with $\sigma = \alpha(n-2)+2$ is considered the threshold for the global existence and finite time blow up for the small data solutions. The aim of this paper is to fill all the case of the exponent and give the classification of the global existence and finite time blow up of the degenerated case of Keller-Segel equation. More specifically, if the exponent satisfies the other condition, then the solution blows up in a finite time for the large initial data.

Theorem 1.4. (Global existence) *Let $\lambda = 1$. For $\alpha > 1$, let (u, ψ) be a weak solution of the degenerated Keller-Segel system (1.9) for $u_0 \geq 0$ obtained in the above Proposition.*
(1) Let $n \geq 2$, If $\alpha > 2 - \frac{2}{n}$, then the solution exists globally in time. Moreover the weak solution satisfies the uniform estimate as follows:

$$\|u(t)\|_\infty \leq C(\|u_0\|_1, \|u_0\|_\infty, \|\Lambda^{-1}u_0\|_2)$$

for $t \in [0, \infty)$.

(2) Let $n \geq 3$. If $\alpha \leq 2 - \frac{2}{n}$ and the initial data is small in the following sense: there exists a constant $C > 0$ such that

$$\|u_0\|_1 \leq C(\|E_n\|_{L_w^{\frac{n}{n-2}}}),$$

where E_n is the fundamental solution of $-\Delta + 1$ in $\mathbb{R}^n$. Then the solution exists globally in time and moreover if $2 - \frac{4}{n+2} < \alpha$ the solution satisfies the uniform boundedness estimate as in above.

Theorem 1.5. (Finite time blow up) *Let (u, ψ) be a weak solution of the degenerated Keller-Segel system (1.9) with $\lambda = 1$ for $u_0 \geq 0$ obtained in the above Proposition. Assuming that $n \geq 3$ and $\alpha \leq 2 - \frac{2}{n}$, and the initial data $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$ with $|x|^2 u_0 \in L^1(\mathbb{R}^n)$ satisfies the following condition:*

$$W(0) \equiv \frac{1}{\alpha - 1}\|u_0\|_\alpha^\alpha - \frac{1}{2}\|\Lambda^{-1}u_0\|_2^2 < 0,$$

where $\Lambda = (-\Delta + 1)^{1/2}$ is the Bessel potential in $\mathbb{R}^n$, then the weak solution does not exists globally in time. Namely there exists $T_m < \infty$ such that for some initial data u_0 the weak solution blows up in a finite time T_m in the following sense;

$$\limsup_{t \to T_m} \|u(t)\|_p = \infty$$

for all $p \in [\alpha, \infty]$.

Remark. For the initial data satisfies the condition in Theorem 1.5, the L^1 norm of the data u_0 is naturally large. Especially for the critical case, $\alpha = 2 - \frac{2}{n}$, the data has to have the large L^1 norm.

For the proof of the local existence of the weak solution, one may adopt the argument of Sugiyama [35] and standard theory of the degenerated parabolic system (as in the theory of p-Laplace heat flow). In fact, the global existence result for the system is heavily depending on the a priori bound for the approximated solutions. One may find the a prori bound by argument from Theorem 1.4.

For bounded domain Ω, the analogous blowing up problem is considered by Biler-Nadzieja-Stanczy [6]. They showed the non-existence of the solution in the bounded domain $\Omega \subset \mathbb{R}^n$ for the Dirichlet boundary condition and the Neumann boundary condition. In those settings, the weighted density $\int_\Omega |x|^2 u(t)dx$ can always make sense and the proof is rather simpler. Analogous result for the Cauchy problem is also considered by Sugiyama [35].

For the semilinear case, $\alpha = 1$, it has already been proved that $n = 2$ is the critical case $\alpha = 1 = 2 - \frac{2}{n}$ and the solution may blow up in finite time for the large initial data. Our theorem is a natural extension of those semilinear case. In particular, the case of the Cauchy problem, the threshold of the global existence and finite time blowing up of the solution is determined by the size of the L^1 norm of the initial data. Especially the semilinear critical case, the threshold number 8π is connected with the best possible constant of the isoperimetric inequality via the Trudinger-Moser type inequality ([27]). The similar result can be also obtained by using the improved Brezis-Merle type inequality (Nagai-Ogawa [25]).

The crucial part of the proof is to show the apriori bound for the weak solution in time globally. It is well understood that the solution of the semilinear equation (1.2) satisfies the following conservation laws:

$$\int_{\mathbb{R}^n} u(t)dx = \int_{\mathbb{R}^n} u_0 dx,$$

$$W(t) + \int_0^t \int_{\mathbb{R}^n} u\,|K(u,\psi)|^2\,dxdt = W(0),$$

with

$$W(t) \equiv \int_{\mathbb{R}^n} (1+u(t))\log(1+u(t))dx - \int_{\mathbb{R}^n} u(t)v(t)\ dx + \frac{1}{2}(\|\nabla\psi(t)\|_2^2 + \|\psi(t)\|_2^2),$$

$$K(u(t),\psi(t)) \equiv \nabla\Big(\log(1+u(t)) - \psi\Big).$$

The global existence part of the weak solution of the degenerated system in the above theorems are essentially depending on the corresponding conservation laws of the quasi-linear case (cf. for the semilinear case [3]).

Finally we discuss about the asymptotic behavior of the global small solution when the degeneracy order is less than the critical case. We denote the weighted Lebesgue space $L^p_a(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n); |\cdot|^a f(\cdot) \in L^p(\mathbb{R}^n)\}$.

Theorem 1.6. (Decay of solution) *Let $1 < \alpha \le 2 - \frac{2}{n}$ and we assume that $u_0 \in L^1_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then the corresponding global weak solution $u(t,x)$ of (1.9) satisfies the following asymptotic behaviors:*
(1) For $1 < \alpha \le 2 - \frac{2}{n}$, if we assume that for some absolute constant $C_n > 0$, $\|u_0\|_1 \le C_n$, then we have

$$\|u(t)\|_p \le C(1+\sigma t)^{-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}, \tag{1.11}$$

where C is only depending on u_0 and n.
(2) For $1 < \alpha < 2 - \frac{2}{n}$, we assume the initial data is small in L^1 sense. Then for $M = \|u_0\|_1$ and for some $\nu > 0$,

$$\|u(t) - MU(t)\|_1 \le C(1+\sigma t)^{-\nu},$$

where $U(t)$ is the Barenbratt solution given by

$$U(t) = (1+\sigma t)^{-n/\sigma} \left(A - \frac{|x|^2}{(1+\sigma t)^{2/\sigma}} \right)_+ ,$$

where $A > 0$ is a constant so that $\|U(t)\|_1 = 1$ and $\sigma = n(\alpha-1)+2$. In particular, the solution $u(t)$ satisfies the uniform decay estimate (1.11).

The semilinear version of the above asymptotic result is obtained by several authors (see for example, Biler-Dolbeault [3] for the case (1.2) and Kozono-Sugiyama [18] for the case (1.1)). The proof of the above asymptotic behavior is depending on the second moment identities. The method developed by Carrillo-Toscani [8] for the Fokker-Planck equation can be applied in our case. There is a variational formulation on the stationary solution of the porous medium equation (see Otto [30]) and the back ground of the proof is lying on this fact. Under the self-similar scaling (cf. Giga-Kohn [13]) we introduce the new scaled variables (t', x') as

$$\begin{cases} t' = \frac{1}{\sigma}\log(1+\sigma t) \\ x' = x/(1+\sigma t)^{1/\sigma} \end{cases}$$

one can find the scaled equation is of the form

$$\begin{cases} \partial_t v - \text{div } (\nabla v^\alpha + xv - e^{-\kappa t} v \nabla \phi) = 0 \qquad t > 0, x \in \mathbb{R}^n, \\ -e^{-2t}\Delta\phi + \lambda\phi = v, \\ \qquad v(0,x) = u_0(x), \end{cases} \tag{1.12}$$

with $\kappa = n + 2 - \sigma = n(2-\alpha)$, where

$$\begin{cases} v(t,x) \equiv e^{nt} u \left(\dfrac{1}{\sigma}(e^{\sigma t} - 1), xe^t \right), \\ \phi(t,x) \equiv e^{nt} \psi \left(\dfrac{1}{\sigma}(e^{\sigma t} - 1), xe^t \right). \end{cases}$$

Then the scaled equation (1.12) also has an analogous entropy; setting

$$\begin{aligned} W_s(v,\phi)(t) &\equiv \frac{1}{2} H(v(t)) - \frac{1}{2} e^{-\kappa t} \int_{\mathbb{R}^n} v(t)\phi(t)\, dx, \\ H(v(t)) &= \frac{2}{\alpha-1}\int_{\mathbb{R}^n} v^\alpha(t) dx + \int_{\mathbb{R}^n} |x|^2 v(t) dx, \\ K_s(x, v(t), \phi(t)) &= K(x,v,\phi) \equiv \nabla\Big(\frac{\alpha}{\alpha-1} v^{\alpha-1} + \frac{1}{2}|x|^2 - e^{-\kappa t}\phi \Big). \end{aligned}$$

the following identities holds in formally (Proposition 4.1):

$$\begin{aligned} &\int_{\mathbb{R}^n} v(t) dx = \int_{\mathbb{R}^n} u_0(x) dx, \\ &W_s(t) + \int_0^t \left[\int_{\mathbb{R}^n} v\, |K(x,v,\phi)(\tau)|^2 \, dx d\tau + e^{-(\kappa+2)\tau} \int_{\mathbb{R}^n} |\nabla\phi(\tau)|^2 dx \right] d\tau = W_s(0). \end{aligned}$$

Hence the decay of the solution follows from the analogous estimate for the global existence of the weak solution. The convergence to the limiting solution is derived from computing the second time derivative of the moment.

This paper is organized as follows. In the following section, we derive the above entropy and free energy bound formally. Based on this conserved quantities, we show the time apriori estimate for the global weak solution in section 2. In section 3, we show the formal blow up proof. The last two sections 4 and 5 are devoted to the proof of the decay of the solution.

2. Free Energy Estimate and Uniform A Priori Bound

2.1. **Conserved quantities.** We start with the following lemma for the conservation law and the entropy functional:

Lemma 2.1. *Let (u, ψ) be a weak solution of (1.9). Then we have the following inequalities:*

$$\int_{\mathbb{R}^n} u(t)dx = \int_{\mathbb{R}^n} u_0(x)dx, \tag{2.1}$$

$$W(t) + \int_0^t \int_{\mathbb{R}^n} u\, |K(u, \psi)|^2\, dxdt \le W(0) \tag{2.2}$$

with

$$\begin{aligned} W(t) &\equiv \frac{1}{\alpha-1}\|u(t)\|_\alpha^\alpha - \frac{1}{2}\int_{\mathbb{R}^n} u(t)\psi(t)\ dx, \\ K(u(t), \psi(t)) &\equiv \nabla\Big(\frac{\alpha}{\alpha-1}u^{\alpha-1} - \psi\Big). \end{aligned} \tag{2.3}$$

Proof of Lemma 2.1. For the completeness, we show the formal proof of those conservation laws. Multiplying (1.9) by $\dfrac{\alpha}{\alpha-1}u^{\alpha-1} - \psi$ and integrate by parts, we see

$$\begin{aligned} \int_{\mathbb{R}^n} \partial_t u\Big(\frac{\alpha}{\alpha-1}u^{\alpha-1} - \psi\Big)\ dx &= -\int_{\mathbb{R}^n} \Big(\nabla u^\alpha - u\cdot\nabla\psi\Big)\cdot\nabla\Big(\frac{\alpha}{\alpha-1}u^{\alpha-1} - \psi\Big)\ dx \\ &= -\int_{\mathbb{R}^n} u\Big|\nabla\Big(\frac{\alpha}{\alpha-1}u^{\alpha-1} - \psi\Big)\Big|^2 dx \end{aligned} \tag{2.4}$$

From the second equation,

$$\int_{\mathbb{R}^n} u\cdot\partial_t\psi dx \quad = \quad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \left(|\nabla\psi|^2 + |\psi|^2\right)\ dx.$$

Thus the left hand side of (2.4) is

$$\frac{d}{dt}\Big(\frac{1}{\alpha-1}\|u(t)\|_\alpha^\alpha - \int_{\mathbb{R}^n} u\psi\ dx + \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla\psi|^2 + |\psi|^2\right) dx\Big), \tag{2.5}$$

Combining (2.4) and (2.5)

$$\frac{d}{dt}W(t) + \int_{\mathbb{R}^n} u\Big|\nabla\Big(\frac{\alpha}{\alpha-1}u^{\alpha-1} - \psi\Big)\Big|^2 dx = 0. \tag{2.6}$$

Integrating in time of both side of (2.6), we obtain the desired estimate. The rigorous justification requires some regularizing argument for the equation in order to escape the degeneracy. ❑

2.2. **Uniformly boundedness.** We only show the a priori estimates for the global existence of the weak solution. The local existence theorem requires some approximation procedures. We do not go into the details in this direction.

Under the condition $\alpha > 2 - \frac{2}{n}$, we show the uniform boundedness of the solution (u, ψ), To see this, the apriori bound for L^α is essential. We start from the following auxiliary lemma

Lemma 2.2. *Let E_n be the fundamental solution of $(-\Delta+1)$ in $\mathbb{R}^n$. Then for $u \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$, let $\psi = E_n u$ be solution of the second equation of the system (1.9), we have*

$$\int_{\mathbb{R}^n} u\psi dx = \frac{1}{2}\left(\|\nabla\psi\|_2^2 + \|\psi\|_2^2\right) \le \|E_n\|_{L_w^{\frac{n}{n-2}}} \|u\|_1^{1-\gamma}\|u\|_\alpha^{1+\gamma}, \tag{2.7}$$

where $\gamma = \frac{\alpha(n-2)}{n(\alpha-1)} - 1 < \alpha - 1$.

Proof of Lemma 2.2. The first identity in (2.7) is obtained directly from the second equation. By the Hölder inequality,

$$\int_{\mathbb{R}^n} u\psi dx \le \|u\|_r \|\psi\|_{r'} \qquad \text{for } \frac{1}{r} + \frac{1}{r'} = 1 \tag{2.8}$$

While

$$\|u(t)\|_r \le \|u\|_1^{1-\gamma}\|u\|_\alpha^\gamma$$

under

$$\frac{1}{r} = 1 - \gamma + \frac{\gamma}{\alpha}$$

and by the Hausdorff-Young inequality,

$$\|\psi\|_{r'} \le \|E_n\|_{L_w^{\frac{n}{n-2}}} \|u\|_\alpha$$

with

$$\frac{1}{r'} = \frac{n-2}{n} + \frac{1}{\alpha} - 1.$$

$\square$

If we wish to choose $\gamma + 1 < \alpha$, then by

$$\gamma\left(1 - \frac{1}{\alpha}\right) = 1 - \frac{1}{r} = \frac{n-2}{n} + \frac{1}{\alpha} - 1,$$

we see

$$\alpha\left(1 - \frac{1}{\alpha}\right) > (\gamma+1)\left(1 - \frac{1}{\alpha}\right) = \frac{n-2}{n}.$$

This relation yields

$$\alpha > 2 - \frac{2}{n}$$

and this condition gives a uniformly boundedness of the L^α norm of the solution.

Proposition 2.3. (1) *Let $\alpha > 2 - \frac{2}{n}$. Then we have*

$$\|u(t)\|_\alpha^\alpha + \frac{1}{2}(\|\nabla\psi(t)\|_2^2 + \|\psi(t)\|_2^2) \le C(W(0) + C_n\|u_0\|_1^{\frac{\alpha(\gamma+1)}{\alpha+\gamma-1}})$$

for all $t \in [0,\infty)$, where $\gamma < \alpha - 1$. In particular,

$$\|u_0\|_\alpha^\alpha \le C(W(0) + C_n\|u_0\|_1^{\frac{\alpha(\gamma+1)}{\alpha+\gamma-1}}).$$

(2) *Let $1 < \alpha \le 2 - \frac{2}{n}$. Then there exists a constant $C = C_n$ which is only depending on n such that for the weak solution satisfying $\|u_0\|_1 \le C_n$, we have*

$$\|u(t)\|_\alpha^\alpha \le C(n, \|u_0\|_1, W(0))$$

for all $t \in [0,\infty)$.

Proof of Proposition 2.3. By the entropy bound (2.2), it suffices to show that

$$\int_{\mathbb{R}^n} u(t)\psi(t)dx$$

is controlled by $\|u(t)\|_\alpha^\alpha$ and $W(0)$. To see this we see by Lemma 2.2 that under the condition $\alpha > 2 - \frac{2}{n}$,

$$\begin{aligned}\int_{\mathbb{R}^n} u(t)\psi(t)dx \le& C\|u(t)\|_1^{1-\gamma}\|u(t)\|_\alpha^{1+\gamma} \\ \le& \varepsilon\|u(t)\|_\alpha^\alpha + C_\varepsilon\|u_0\|_1^{\frac{\alpha(1-\gamma)}{\alpha-(\gamma+1)}}\end{aligned} \tag{2.9}$$

and the desired estimate follows from (2.9), L^1 conservation law and entropy bound $W(t) \le W(0)$.

For the case $1 < \alpha \le 2 - \frac{2}{n}$, we again use Lemma 2.2 and the entropy bound (2.2), we see

$$\frac{2}{\alpha-1}\|u(t)\|_\alpha^\alpha \le W(0) + \|E_n\|_{L_w^{n/(n-2)}}\|u_0\|_1^{1-\gamma}\|u(t)\|_\alpha^{\gamma+1}.$$

When $\alpha = 2 - \frac{2}{n}$ then $\gamma + 1 = \alpha$ and the smallness condition

$$\|u_0\|_1^{1-\gamma} < \frac{2}{\alpha-1}\|E_n\|_{L_w^{n/(n-2)}}^{-1}$$

directly gives the uniform boundedness of $\|u(t)\|_\alpha$. For $1 < \alpha < 2 - \frac{2}{n}$, there exists a constant C_0 which is determined by $W(0)$, α, n and $\|E_n\|_{L_w^{n/(n-2)}}$ such that for $\|u_0\|_1 \le C_0$ then we also see that

$$\|u(t)\|_\alpha^\alpha \le C(n, \alpha, \|u_0\|_1, W(0))$$

uniformly in t. ❑

Remark The exponent of L^1 norm of the right hand side of the proposition satisfies

$$\frac{\alpha(1-\gamma)}{\alpha-(\gamma+1)} \ge 2.$$

Theorem 2.4. *Let $n \ge 2$ and suppose that $\alpha > 1$. Then under the condition that $\|u(t)\|_\alpha$ is uniformly bounded in t, we have for any $t > 0$,*

$$\|u(t)\|_\infty + \|\psi(t)\|_\infty \le C(W(0), \|u_0\|_1, \|u_0\|_\infty, \alpha, n).$$

Hence the weak solution globally exists.

Proof of Theorem 2.4. Firstly we observe that for some $r_0 > n$, we have the uniform bounded estimate for $\|u(t)\|_{r_0}$ by Proposition 2.3. We apply the standard parabolic estimate and we see for any $r > \alpha$ that

$$\frac{d}{dt}\|u(t)\|_r^r + \frac{2r}{\alpha+r-1}\|\nabla u^\gamma(t)\|_2^2 = \int_{\mathbb{R}^n} \nabla u^r(t)\cdot\nabla\psi(t)dx \le \|u(t)\|_{r+1}^{r+1}, \tag{2.10}$$

where $\gamma = \frac{1}{2}(\alpha + r - 1)$. Now we invoke the Gagliardo-Nirenberg interpolation inequality

$$\begin{aligned}&\|f\|_{(r+1)/\gamma} \le C\|f\|_{r/\chi\gamma}^{1-\sigma}\|\nabla f\|_2^\sigma, \\ &\frac{\gamma}{r+1} = \frac{\chi\gamma(1-\sigma)}{r} + \sigma\left(\frac{1}{2} - \frac{1}{n}\right)\end{aligned} \tag{2.11}$$

for some $\chi > 1$. It follows by substituting $f = u^\gamma(t)$,

$$\|u(t)\|_{r+1}^{r+1} \le C\|u(t)\|_{r/\chi}^{(r+1)(1-\sigma)}\|\nabla u^\gamma(t)\|_2^{\sigma(r+1)/\gamma}.$$

If we assume that $\sigma(r+1)/\gamma < 2$ which is assured under the condition

$$2 - \frac{2r}{\chi n} < \alpha, \tag{2.12}$$

we have

$$\|u(t)\|_{r+1}^{r+1} \le C\|u(t)\|_{r/\chi}^{\beta} + \|\nabla u^\gamma(t)\|_2^2 \tag{2.13}$$

with

$$\beta = (1-\sigma)(r+1)\left(1 - \frac{2\gamma}{\sigma(r+1)}\right)^{-1}.$$

Similarly we see that

$$\begin{aligned} \|u(t)\|_r^r \le& C\|u(t)\|_1^{r(1-\mu)}\|\nabla u^\gamma(t)\|_2^{r\mu/\gamma} \\ \le& C\|u(t)\|_1^{\delta} + \|\nabla u^\gamma(t)\|_2^2, \end{aligned} \tag{2.14}$$

where $r\mu/\gamma < 2$ under $1 - \frac{r}{n} < \alpha$ and

$$\begin{aligned} \delta =&(1-\mu)r\left(1 - \frac{r\mu}{2\gamma}\right)^{-1} \\ =&\frac{(1-\mu)(1+(\alpha-1)/r)}{1-\mu+(\alpha-1)/r}r \\ =&\frac{1-\mu+(1-\mu)(\alpha-1)/r}{1-\mu+(\alpha-1)/r}r < r, \end{aligned}$$

since $(1-\mu)(\alpha-1)/r < (\alpha-1)/r$. Thus combining (2.10), (2.13) and (2.14), we obtain

$$\frac{d}{dt}\|u(t)\|_r^r + C_0\|u(t)\|_r^r \le C\|u(t)\|_{r/\chi}^{\beta} + C\|u(t)\|_1^{\delta} \tag{2.15}$$

with

$$\beta =(1-\sigma)(1+r)\left(1 - \frac{\sigma(r+1)}{2\gamma}\right)^{-1}.$$

Let us firstly choose that $\chi = r/\alpha > 1$. Then we see from (2.15) that

$$\frac{d}{dt}\|u(t)\|_{r_0}^{r_0} + C_0\|u(t)\|_{r_0}^{r_0} \le C\|u(t)\|_{\alpha}^{\beta} + C\|u(t)\|_1^{\delta},$$

under the condition that $2 - \frac{4}{n+2} < \alpha$. Since by the assumption, the right hand side is uniformly bounded in t, it follows that by multiplying $e^{C_0 t}$ into both side of the inequality, we see that

$$\begin{aligned} \|u(t)\|_{r_0}^{r_0} \le&\|u_0\|_{r_0}^{r_0} + \left(C\sup_{t\in[0,T]}\|u(t)\|_{\alpha}^{\beta} + C\|u_0\|_1^{\delta}\right)\int_0^t e^{-C_0(t-s)}ds \\ \le&\|u_0\|_{r_0}^{r_0} + C\|u_0\|_1^{\delta} + C\sup_{t\in[0,T]}\|u(t)\|_{\alpha}^{\beta}. \end{aligned} \tag{2.16}$$

The above estimate combining the Sobolev inequality and the elliptic estimate implies

$$\|\nabla\psi(t)\|_\infty \le C\|u(t)\|_{r_0}$$

for some $r_0 > n$, where C is independent of t, implies the uniform bound for $\|\nabla\psi(t)\|_\infty$.

Now we show for the general case $r \in [r_0, \infty]$. Starting from the L^r inequality (2.10),

$$
\begin{aligned}
\frac{d}{dt}\|u(t)\|_r^r + \frac{2r}{\alpha+r-1}\|\nabla u^\gamma(t)\|_2^2 \le& \|\nabla\psi(t)\|_\infty \int_{\mathbb{R}^n} |\nabla u^r| dx \\
\le& \frac{2r\|\nabla\psi\|_\infty}{r+\alpha-1}\|u(t)\|_{r-\alpha+1}^{(r-\alpha+1)/2}\|\nabla u^\gamma\|_2,
\end{aligned}
\tag{2.17}
$$

where $\gamma = \frac{1}{2}(\alpha+r-1)$. Analogous to (2.11), the Gagliardo-Nirenberg interpolation inequality

$$
\begin{aligned}
&\|f\|_{(r+1-\alpha)/\gamma} \le C\|f\|_{r/\chi\gamma}^{1-\sigma}\|\nabla f\|_2^\sigma, \\
&\frac{\gamma}{r+1-\alpha} = \frac{\chi\gamma(1-\sigma)}{r} + \sigma\left(\frac{1}{2}-\frac{1}{n}\right)
\end{aligned}
$$

for some $\chi > 1$. It then follows by substituting $f = u^\gamma(t)$,

$$
\|u(t)\|_{r+1-\alpha}^{r+1-\alpha} \le C\|u(t)\|_{r/\chi}^{(r+1-\alpha)(1-\sigma)}\|\nabla u^\gamma(t)\|_2^{\sigma(r+1-\alpha)/\gamma}. \tag{2.18}
$$

If we assume that $\sigma(r+1-\alpha)/\gamma < 2$ which is assured under

$$
1 - \frac{r}{n} < \alpha,
$$

we have

$$
\|u(t)\|_{r+1-\alpha}^{r+1-\alpha} \le C\|u(t)\|_{r/2}^\beta + \|\nabla u^\gamma(t)\|_2^2,
$$

where

$$
\begin{aligned}
\beta =& (1-\sigma)(1+r-\alpha)\left(1-\frac{\sigma(r+1-\alpha)}{2\gamma}\right)^{-1} \\
=& \frac{2\gamma(1-\sigma)}{2\gamma/(r+1-\alpha)-\sigma} \\
=& r\frac{1-\sigma+(1-\sigma)(\alpha-1)/r}{1-\sigma+2(\alpha-1)/(r+1-\alpha)} < r.
\end{aligned}
$$

Thus we again use (2.10), (2.15) that

$$
\frac{d}{dt}\|u(t)\|_r^r + C_0\|u(t)\|_r^r \le C\|u(t)\|_{r/\chi}^\beta + C\|u(t)\|_1^\delta,
$$

under the condition $2 - \frac{4}{n+2} < \alpha$. Note that all the constants appearing the above inequality is depending on r but they can be chosen uniformly bounded as $r \to \infty$. It follows that by multiplying $e^{C_0 t}$ into both side of the inequality, we see that

$$
\|u(t)\|_r^r \le \|u_0\|_r^r + \left(C\sup_{t\in[0,T]}\|u(t)\|_{r/\chi}^\beta + C\|u_0\|_1^\delta\right)\int_0^t e^{-C_0(t-s)}ds.
$$

For sufficiently large $r > n$, we see that

$$
\|u(t)\|_r + M \le C^{1/r}(M + \sup_{t\in[0,T]}\|u(t)\|_{r/\chi})
$$

for all $t \in [0,T]$, where $M = \max(\|u_0\|_1, \|u_0\|_\infty)$. Now choosing $r = \chi^m$, we see by Nash-Moser iteration argument that

$$
\|u(t)\|_\infty \le C^{c\sum_m \chi^{-m}}(M + \sup_{t\in[0,T]}\|u(t)\|_{r_0}).
$$

This combining with the estimate (2.16) yields the desired uniform estimate under the uniform bound of $\|u(t)\|_\alpha$.

For the case $1 < \alpha \le 2 - \frac{4}{n+2}$, we draw back to the estimate (2.15):

$$\frac{d}{dt}\|u(t)\|_r^r + C_0\|u(t)\|_r^r \le C\|u(t)\|_{r/\chi}^{\beta} + C\|u(t)\|_1^{\delta} \tag{2.19}$$

with

$$\beta = \frac{2\gamma(1-\sigma)}{2\gamma/(r+1)-\sigma} = r\frac{1-\sigma+(1-\sigma)(\alpha-1)/r}{1-\sigma+(\alpha-2)/(r+1)} \equiv r\mu_r.$$

Thus it follows an analogous estimate as in (2.2) with μ_r may be larger than 1. Let $r = r_0 > n$ is fixed and we choose $\chi > 1$ properly so that by finite times iteration,

$$\|u(t)\|_{r_0} \le C^{c\sum_{k=1}^m \chi^{-m}}(M + \sup_{t\in[0,T]}\|u(t)\|_{r_0/\chi^m})^{c\prod_{k=1}^m \mu_{\chi^k}},$$

where the exponent r/χ^m reaches in $[1,\alpha]$ and hence we have

$$\|u(t)\|_{r_0} \le C \text{ independent of } t.$$

The rest of the argument is similar to the case $2 - \frac{4}{n+2} < \alpha$.

Note that we can eliminate the initial restriction $\|u_0\|_r$ by the parabolic regularity argument. The regularity of $\psi(t)$ immediately follows from the standard elliptic estimate for the second equation. ❑

Lemma 2.5. *Let $\alpha > 1$. For any $f \in L^\alpha(\mathbb{R}^n)$ with $|x|^2u_0(x) \in L^1(\mathbb{R}^n)$ then we have*

$$\|f\|_1 \le C\left(\int_{\mathbb{R}^n}|x|^2|f(x)|dx\right)^{1-\kappa}\|f\|_\alpha^\kappa,$$

where

$$\kappa = \frac{2\alpha}{\alpha(n+2)-n}.$$

Proof of Lemma 2.5. For simplicity we assume $f \ge 0$. The general case can be easily obtained by a simple modification. For $r > 0$ chosen to be later, we see for some constants a, $b > 0$,

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)dx \le& \int_{B_r} f(x)dx + \int_{B_r^c} f(x)dx\\ \le& ar^{n/\alpha'}\left(\int_{B_r}|f(x)|^\alpha dx\right)^{1/\alpha} + \frac{b}{r^2}\int_{B_r^c}|x|^2 f(x)dx\\ \le& ar^{n/\alpha'}\|f\|_\alpha + \frac{b}{r^2}\int_{\mathbb{R}^n}|x|^2 f(x)dx\\ \equiv& Ar^{n(\alpha-1)/\alpha} + Br^{-2} \equiv f(r).\end{aligned} \tag{2.20}$$

Then $f'(r) = \frac{n}{\alpha}Ar^{n/\alpha-1} - 2Br^{-3} = 0$ gives

$$r^{\frac{n}{\alpha'}+2} = c\frac{B}{A}.$$

Thus

$$r = c\left(\frac{B}{A}\right)^{\frac{\alpha}{\alpha(n+2)-n}}$$

and desired inequality follows. ❑ ❑

3. Finite Time Blow-up

3.1. **Dimension Analysis.** Let $\lambda > 0$ and $\mu > 0$ be a scaling parameter. We introduce the following scaled solutions:

$$\begin{cases} u_{\lambda,\mu} = \lambda u(\mu x), \\ \psi_{\lambda,\mu} = \lambda\mu^{-2}\psi(\mu x). \end{cases} \tag{3.1}$$

A direct computation gives

Lemma 3.1. *By the scaling we see*

$$\begin{aligned} \|u_{\lambda,\mu}\|_1 =& \lambda\mu^{-n}\|u\|_1, \\ \|u_\lambda\|_\alpha^\alpha =& \lambda^\alpha\mu^{-n}\|u\|_\alpha^\alpha, \\ \int_{\mathbb{R}^n} u_\lambda\psi_\lambda dx =& \lambda^2\langle\mu\rangle^{-(n+2)}\int_{\mathbb{R}^n} u\psi dx. \end{aligned} \tag{3.2}$$

Now we see if the initial entropy may be chosen as negative.

Lemma 3.2. *Let $n \geq 2$. and $u \in L^1 \cap L^\alpha$. For λ, $\mu > 0$ we define the scaled function $u_{\lambda,\mu} = \lambda u(\mu x)$. Set $\|u_{\alpha,\mu}\|_1 = A$,, then for $\alpha < 2 - \frac{2}{n}$, by choosing $\lambda >> 1$ large enough then*

$$W(0) = \frac{1}{\alpha-1}\|u_{\lambda,\mu}\|_\alpha^\alpha - \frac{1}{2}\int_{\mathbb{R}^n} u_{\lambda,\mu}(-\Delta+1)^{-1}u_{\lambda,\mu}dx < 0.$$

If $\alpha = 2 - \frac{2}{n}$ then by choosing A sufficiently large, we have the same conclusion.

Proof of Lemma 3.2. By $\|u_{\lambda,\mu}\|_1 = \lambda\mu^{-n} = A$, we have $\mu = (\lambda/A)^{1/n}$. Then we have for positive constants $B = \frac{\|u\|_\alpha^\alpha}{\alpha-1}$, $C = \frac{1}{2}\|\Lambda^{-1}u\|_2^2$,

$$\begin{aligned} W(u_{\lambda,\mu}) =& \frac{1}{\alpha-1}\lambda^\alpha\mu^{-n}\|u\|_\alpha^\alpha - \frac{1}{2}\lambda^2\mu^{-n}\langle\mu\rangle^{-2}\int_{\mathbb{R}^n} u(-\Delta+1)^{-1}udx \\ =& AB\lambda^{\alpha-1} - C\frac{\lambda^2 A^{\frac{n+2}{n}}}{\lambda(A^{\frac{1}{n}}+\lambda^{\frac{1}{n}})^2} \\ =& A\lambda^{\alpha-1}\left(B - \frac{\lambda^{2-\alpha}A^{\frac{2}{n}}C}{(A^{\frac{1}{n}}+\lambda^{\frac{1}{n}})^2}\right) \\ =& \begin{cases} A\lambda^{\alpha-1}\left(B - A^{2/n}C\lambda^{2-\frac{2}{n}-\alpha}\right), & \lambda > 1, \\ A\lambda^{\alpha-1}\left(B - C\lambda^{2-\alpha}\right), & \lambda < 1. \end{cases} \end{aligned} \tag{3.3}$$

Hence when $\alpha < 2 - \frac{2}{n}$ then by choosing λ large, we have $W < 0$ under the condition. If $\alpha = 2 - \frac{2}{n}$ then choose $A = \|u\|_1$ sufficiently large, then we have $W < 0$.

□

3.2. **Virial Law and Blow-up.** In this section, we show the non-existence of the weak solution and finite time blow up by a formal way. The argument is almost similar to the one in [6] (cf. [29]).

Lemma 3.3. *Let (u, ψ) be a weak solution of (1.9). Then it follows that*

$$\frac{d}{dt}\int_{\mathbb{R}^n} |x|^2 u(t)dx = 2n\|u(t)\|_\alpha^\alpha + 2\int_{\mathbb{R}^n} xu(t)\cdot\nabla\psi(t)dx. \tag{3.4}$$

Proof of Lemma 3.3. Multiplying the equation by $|x|^2$ and integrate it by parts. We obtain it $\square$

Here we show the rough result on the finite time blow up.

Theorem 3.4 ([6]). *Let $n \geq 3$ and $1 \leq \alpha \leq 2 - \frac{2}{n}$. Then for $u_0 \in L^1 \cap L^\alpha$ with $|x|^2 u_0(x) \in L^1(\mathbb{R}^n)$ and*

$$W(0) \equiv \frac{1}{\alpha - 1}\|u_0\|_\alpha^\alpha - \frac{1}{2}\|(-\Delta + 1)^{-1/2} u_0\|_2^2 < 0. \tag{3.5}$$

The corresponding weak solution obtained in Proposition 1.3 blows up in a finite time.

Proof of Theorem 3.4. The proof is essentially similar to the one in [6]. We only give the formal observation. First we see from Lemma 3.3 that

$$\frac{d}{dt}\int_{\mathbb{R}^n} |x|^2 u(t)dx = 2n\|u(t)\|_\alpha^\alpha + 2\int_{\mathbb{R}^n} x u(t) \cdot \nabla\psi(t)dx. \tag{3.6}$$

Next we invoke the Pokhozaev identity for the second equation. We multiply the elliptic part of the system by the generator of the dilation $x \cdot \nabla\psi$ and integrate it by parts, it follows

$$\begin{aligned}
\int_{\mathbb{R}^n} x \cdot \nabla\psi(t) u(t)dx &= \int_{\mathbb{R}^n} \nabla_i\psi(t)\left(\delta_{ij}\nabla_j\psi(t) + x \cdot \nabla\nabla_i\psi(t)\right)dx + \frac{1}{2}\int_{\mathbb{R}^n} x \cdot \nabla|\psi(t)|^2 dx \\
&= \left(1 - \frac{n}{2}\right)\int_{\mathbb{R}^n} |\nabla\psi(t)|^2 dx - \frac{n}{2}\int_{\mathbb{R}^n} |\psi(t)|^2 dx \\
&= \left(1 - \frac{n}{2}\right)\int_{\mathbb{R}^n} u(t)\psi(t)dx - \|\psi(t)\|_2^2.
\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^n} |x|^2 u(t)dx =& 2n\|u(t)\|_\alpha^\alpha + (2-n)\int_{\mathbb{R}^n} u(t)\psi(t)dx - 2\|\psi(t)\|_2^2 \\
=& 2(n-2)\left(\frac{1}{\alpha - 1}\|u(t)\|_\alpha^\alpha - \frac{1}{2}\int_{\mathbb{R}^n} u(t)\psi(t)dx\right) \\
&+ \left(2n - \frac{2(n-2)}{\alpha - 1}\right)\|u(t)\|_\alpha^\alpha - 2\|\psi\|_2^2 \\
=& 2(n-2)W(t) + 2n\left(\frac{\alpha - 2 + \frac{2}{n}}{\alpha - 1}\right)\|u(t)\|_\alpha^\alpha - 2\|\psi\|_2^2.
\end{aligned}$$

Hence by assuming $n \geq 3$ and $\alpha \leq 2 - \frac{2}{n}$ then it is possible to choose the initial data such as $W(0) < 0$ by Lemma 3.1 and we see

$$\frac{d}{dt}\int_{\mathbb{R}^n} |x|^2 u(t)dx \leq W(0) < 0, \tag{3.8}$$

which yields a contradiction within a finite time. $\square$

4. Time Decay of Small Solution

In this section, we consider the decay and asymptotic behavior of the global weak solution of the degenerated Keller-Segel system.

4.1. **Rescaled equation.** To avoid the confusion, we change the notation slightly.

$$\begin{cases} \partial_t u - \Delta u^\alpha + \nabla(u\nabla\psi) = 0, \quad x \in \mathbb{R}^n,\ t > 0 \\ -\Delta\psi + \lambda\psi = u \quad x \in \mathbb{R}^n,\ t > 0 \\ \qquad u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{4.1}$$

We introduce the new scaled variables (t', x') as

$$\begin{cases} t' = \frac{1}{\sigma}\log(1+\sigma t) \\ x' = x/(1+\sigma t)^{1/\sigma} \end{cases} \tag{4.2}$$

and introduce the new scaled unknown function $u(t', x')$ by

In regarding to the presence of $\lambda > 0$, we may choose the scaling that maintain the λ term as it is. Namely we let

$$u(t,x) = (1+\sigma t)^{-n/\sigma} v\left(\frac{1}{\sigma}\log(1+\sigma t), \frac{x}{(1+\sigma t)^{1/\sigma}}\right)$$
$$\psi(t,x) = (1+\sigma t)^{-n/\sigma} \phi\left(\frac{1}{\sigma}\log(1+\sigma t), \frac{x}{(1+\sigma t)^{1/\sigma}}\right).$$

Or it may be written as

$$v\left(t', x'\right) \equiv e^{nt'} u\left(\frac{1}{\sigma}(e^{\sigma t'} - 1), x' e^{t'}\right)$$
$$\phi\left(t', x'\right) \equiv e^{nt'} \psi\left(\frac{1}{\sigma}(e^{\sigma t'} - 1), x' e^{t'}\right)$$

and the resulting scaling equation of (v, ϕ) follows by setting $\kappa = n + 2 - \sigma = n(2 - \alpha)$,

$$\begin{cases} \partial_t v - \text{div }(\nabla v^\alpha + xv - e^{-\kappa t} v\nabla\phi) = 0 \qquad t > 0, x \in \mathbb{R}^n, \\ -e^{-2t}\Delta\phi + \lambda\phi = v, \\ \qquad v(0,x) = u_0(x), \end{cases} \tag{4.3}$$

In this case, the vanishing exponent as before can be found as $\alpha = 2$ by

$$0 = \sigma - n - 2 = n(\alpha - 2)$$

and thus the subcritical case is corresponding to $\alpha < 2$. Hereafter we analyze the above rescaled equation (4.3) to see the asymptotic behavior of the solution. We slightly change the outlook of the solution as follows:

The existence of the weak solution of (4.3) can be proven by a similar way to the original equation. Indeed the scaling does not change any analytical feature of the original weak solution so that the solution can be obtained from the weak solution of (1.9). Namely we again consider the nonnegative weak solution $v(t, x)$ as before.

4.2. **Rescaled Conservations of Mass, Entropy and Moment.** We revisited to the conservation laws and the entropy functional for the rescaled equation (4.3):

Proposition 4.1. *Let $\kappa = n(2-\alpha) > 0$ and assume that the initial data $u_0 \in L^\alpha(\mathbb{R}^n) \cap L^1_2(\mathbb{R}^n)$ with $u_0 \geq 0$. Let (v, ϕ) be a weak solution of (4.3) and set the functionals $W_s(v, \phi)$, $H(v(t))$ and*

$K_s(v,\phi)$ as follows:

$$
\begin{aligned}
&W_s(v,\phi)(t) \equiv \frac{1}{2}H(v(t)) - \frac{1}{2}e^{-\kappa t}\int_{\mathbb{R}^n} v(t)\phi(t)\ dx,\\
&H(v(t)) = \frac{2}{\alpha-1}\int_{\mathbb{R}^n} v^{\alpha}(t)dx + \int_{\mathbb{R}^n}|x|^2 v(t)dx,\\
&K_s(x,v(t),\phi(t)) \equiv \nabla\Big(\frac{\alpha}{\alpha-1}v^{\alpha-1} + \frac{1}{2}|x|^2 - e^{-\kappa t}\phi\Big).
\end{aligned}
\tag{4.4}
$$

Then we have the following identities:

$$
\begin{aligned}
&\int_{\mathbb{R}^n} v(t)dx = \int_{\mathbb{R}^n} u_0(x)dx,\\
&W_s(t) + \int_0^t \Big[\int_{\mathbb{R}^n} v\,|K_s(x,v,\phi)(\tau)|^2\,dxd\tau + e^{-(\kappa+2)\tau}\int_{\mathbb{R}^n}|\nabla\phi(\tau)|^2dx\Big]\,d\tau \le W_s(0).
\end{aligned}
\tag{4.5}
$$

Proof of Proposition 4.1. Let $\kappa = -(\sigma - n - 2) = n(2-\alpha) > 0$. Multiplying (1.9) by $\dfrac{\alpha}{\alpha-1}v^{\alpha-1} + \dfrac{1}{2}|x|^2 - e^{-\kappa t}\phi$ and integrating by parts, we see that

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\partial_t v\Big(\frac{\alpha}{\alpha-1}v^{\alpha-1} + \frac{1}{2}|x|^2 - e^{-\kappa t}\phi\Big)\ dx\\
&\quad= -\int_{\mathbb{R}^n} v\Big|\nabla\Big(\frac{\alpha}{\alpha-1}v^{\alpha-1} + \frac{1}{2}|x|^2 - e^{-\kappa t}\phi\Big)\Big|^2dx.
\end{aligned}
\tag{4.6}
$$

While the left hand side can be treated as;

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\partial_t v\cdot\Big(\frac{\alpha}{\alpha-1}v^{\alpha-1} + \frac{1}{2}|x|^2 - e^{-\kappa t}\phi\Big)\ dx\\
=&\frac{d}{dt}\Big[\frac{1}{\alpha-1}\int_{\mathbb{R}^n} v^{\alpha}\ dx + \frac{1}{2}\int_{\mathbb{R}^n}|x|^2 v dx - \int_{\mathbb{R}^n} e^{-\kappa t}v\phi\ dx\Big] + \int_{\mathbb{R}^n} v\cdot\partial_t(e^{-\kappa t}\phi)\ dx.
\end{aligned}
$$

By the elliptic part of the system,

$$
\begin{aligned}
\int_{\mathbb{R}^n} v\cdot\partial_t\phi dx =& e^{-2t}\int\nabla\phi\cdot\nabla\partial_t\phi dx + \lambda\int_{\mathbb{R}^n}\phi\cdot\partial_t\phi\ dx\\
=&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\big(e^{-2t}|\nabla\phi|^2 + \lambda|\phi|^2\big)\ dx + e^{-2t}\int_{\mathbb{R}^n}|\nabla\phi|^2dx.
\end{aligned}
$$

Namely we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n} v\cdot\partial_t(e^{-\kappa t}\phi)dx =& \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} e^{-\kappa t}v\phi dx + \kappa e^{-\kappa t}\int_{\mathbb{R}^n} v\phi dx\\
&+ e^{-(\kappa+2)t}\int_{\mathbb{R}^n}|\nabla\phi|^2dx - \kappa e^{-\kappa t}\int_{\mathbb{R}^n} v\phi dx.
\end{aligned}
$$

Thus the left hand side of (4.6) is

(4.7)

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{\alpha-1}\|v(t)\|_{\alpha}^{\alpha} + \frac{1}{2}\int_{\mathbb{R}^n}|x|^2 v dx - \int_{\mathbb{R}^n} e^{-\kappa t}v\phi\ dx + \frac{1}{2}\int_{\mathbb{R}^n} e^{-\kappa t}v\phi dx\Big) + e^{-(\kappa+2)t}\int_{\mathbb{R}^n}|\nabla\phi|^2dx\\
=&\frac{d}{dt}W_s(v,\phi)(t) + e^{-(\kappa+2)t}\int_{\mathbb{R}^n}|\nabla\phi|^2dx.
\end{aligned}
$$

Combining (4.6) and (4.7),

(4.8)
$$W_s(v,\phi)(t)+\int_0^t\left[\int_{\mathbb{R}^n} v(\tau)\Big|K_s(x,v(\tau),\phi(\tau))\Big|^2dx+e^{-(\kappa+2)\tau}\int_{\mathbb{R}^n}|\nabla\phi(\tau)|^2dx\right]d\tau=W_s(u_0)$$

Again the rigorous justification requires regularizing argument for the equation and we obtain the inequality version of (4.8) as a consequence. ❑

The following estimate is a direct consequence of the above a priori bound of the rescaled solution.

Proposition 4.2. *Let $(v(t),\phi(t))$ be a weak solution of (4.3). For the case $1<\alpha\le 2-\frac{2}{n}$ with small data*
$$\|u_0\|_1\le C_n.$$
(1) Then we have
$$\|v(t)\|_q\le C$$
for all $1\le q\le\infty$ and
(2) for all $n/n-1<r\le\infty$,
$$\|\nabla\phi(t)\|_r\le Ce^{2t}.$$

Proof of Proposition 4.2.

The proof goes in a similar way of proof of Lemma 2.2. Let $E_{\lambda,t}$ be the fundamental solution of $(-e^{-2t}\Delta+\lambda)$ in $\mathbb{R}^n$. Then for $v\in L^1(\mathbb{R}^n)\cap L^\alpha(\mathbb{R}^n)$, let $\phi=E_{\lambda,t}*v$ be solution of the second equation of the system (4.3). Then we have

$$\frac{e^{-\kappa t}}{2}\left(e^{-2t}\|\nabla\phi\|_2^2+\lambda\|\phi\|_2^2\right)=\int_{\mathbb{R}^n}e^{-\kappa t}v\phi dx\le e^{-\kappa t}\|E_{\lambda,t}\|_{L_w^{\frac{n}{n-2}}}\|v\|_1^{1-\gamma}\|v(t)\|_\alpha^{1+\gamma}\tag{4.9}$$

for any $\gamma=\frac{\alpha(n-2)}{n(\alpha-1)}-1<\alpha-1$.

Indeed, by the Hölder inequality,
$$\begin{aligned}\int_{\mathbb{R}^n}v\phi dx\le&\|v\|_r\|\phi\|_{r'}\qquad\text{for }\frac{1}{r}+\frac{1}{r'}=1\\ \le&\|v\|_1^{1-\gamma}\|v\|_\alpha^\gamma\|\phi\|_{r'}\qquad\left(\frac{1}{r}=1-\gamma+\frac{\gamma}{\alpha}\right)\\ \le&\|E_{\lambda,t}\|_{L_w^{\frac{n}{n-2}}}\|v\|_1^{1-\gamma}\|v\|_\alpha^{1+\gamma}\qquad\left(\frac{1}{r'}=\frac{n-2}{n}+\frac{1}{\alpha}-1\right).\end{aligned}\tag{4.10}$$

Under the assumption $\alpha\le 2-\frac{2}{n}$,
$$\alpha\left(1-\frac{1}{\alpha}\right)\le(\gamma+1)\left(1-\frac{1}{\alpha}\right)=\frac{n-2}{n}$$
and this gives $\gamma+1\ge\alpha$. Hence noting that $\|E_{\lambda,t}\|_{n/(n-2)}\simeq Ce^{2t}\le Ce^{\kappa t}$, we have

$$e^{-\kappa t}\int_{\mathbb{R}^n}v(t)\phi(t)dx\le C\|v(t)\|_1^{1-\gamma}\|v(t)\|_\alpha^{1+\gamma}\tag{4.11}$$

and
$$\|v(t)\|_\alpha^\alpha\le W_s(0)+C\|u_0\|_1^{1-\gamma}\|v(t)\|_\alpha^{1+\gamma},$$

where $\gamma+1\ge\alpha$. Therefore under the smallness condition

$$\|u_0\|_1 < C_n,$$

we reach the bound

$$\|v(t)\|_\alpha^\alpha \le C$$

uniformly in t. Hence for $1\le q\le 2-\frac{2}{n}$, the estimate (4.11), L^1 conservation law and the entropy bound

$$W_s(t)\le W_s(0)$$

imply

$$\frac{1}{2(\alpha-1)}\|v(t)\|_\alpha^\alpha+\frac{1}{2}\int_{\mathbb{R}^n}|x|^2 v(t)dx \le C(W_s(0)+C_n\|u_0\|_1^{\frac{\alpha(1-\gamma))}{\alpha-(\gamma+1)}}) \tag{4.12}$$

for all $t\in[0,\infty)$, where $\gamma<\alpha-1$. Here we note that

$$\frac{\alpha(1-\gamma)}{\alpha-(1+\gamma)}=\frac{n(\alpha-2)+2\alpha}{n\alpha+2}.$$

For the case $q\ge 2-\frac{2}{n}$, the estimate is quite similar to the proof of Theorem 2.4. We apply the standard parabolic estimate that we see for any $q>\alpha$ that

$$\begin{aligned}&\frac{d}{dt}\|v(t)\|_q^q+n(q-1)\|v(t)\|_q^q+\frac{2r}{\alpha+q-1}\|\nabla v^\gamma(t)\|_2^2\\ =&(q-1)e^{-\kappa t}\int_{\mathbb{R}^n}\nabla v^r(t)\cdot\nabla\phi(t)dx\le C(q-1)e^{-(\kappa-2)t}\|v(t)\|_{q+1}^{q+1}\end{aligned} \tag{4.13}$$

by the positivity of (v,ϕ), where $\gamma=\frac{1}{2}(\alpha+q-1)$. Noting $\kappa>2$ under $1<\alpha<2-\frac{2}{n}$, the very much similar argument in the proof of Theorem 2.4 implies

$$\|v(t)\|_q\le C(W_s(0),\|u_0\|_1,\|u_0\|_\infty)$$

for any $1\le q\le\infty$ and we obtain the desired apriori estimate for $t\in[0,\infty)$. Note that one can eliminate the initial restriction $\|u_0\|_q$ by the parabolic regularity argument.

The estimate for the potential term ϕ directly follows from the estimate for $v(t)$ and the Hardy-Littlewood inequality: by $\nabla\phi=(-e^{-2t}\Delta+1)^{-1}\nabla v$,

$$\begin{aligned}\|\nabla\phi\|_q=&\|(-e^{-2t}\Delta+1)^{-1}\nabla v\|_q\\ \le& Ce^{2t}\|(-e^{-2t}\Delta+1)^{-1}e^{-2t}\Delta v\|_r\\ \le& Ce^{2t}\|v\|_r\le Ce^{2t}\end{aligned}$$

with $q>n/(n-1)$ and

$$\frac{1}{q}=\frac{1}{r}-\frac{1}{n}.$$

$\square$

Once we obtain the above uniform bound for the rescaled solution, we can immediately obtain the time decay estimate for the solution of the original equation.

$$\int_{\mathbb{R}^n}v^q(t',x')dx'=\int_{\mathbb{R}^n}e^{n(q-1)t'}u^q(t,x)dx=(1+\sigma t)^{(q-1)n/\sigma}\int_{\mathbb{R}^n}u^q(t,x)dx \tag{4.14}$$

in the original variables (t,x). Hence we obtain the following decay estimate for the original solution as a corollary of Proposition 4.2.

Proposition 4.3. *Let $u_0 \in L^1_2(\mathbb{R}^n) \cap L^\infty$ and $(u(t), \psi(t))$ be a weak solution of (1.9). Then for $1 < \alpha \leq 2 - \frac{2}{n}$ with small initial data $\|u_0\|_1 < \varepsilon$, we have*

$$\|u(t)\|_q \leq C(1+\sigma t)^{-\frac{n}{\sigma}(1-\frac{1}{q})}$$

for all $1 \leq q \leq \infty$.

5. Asymptotic Profile

The expected asymptotic profile of the decaying solution is governed by the principal term and it is corresponding to the Barenbladt solution of the single porous medium equation

$$\partial_t u - \Delta u^\alpha = 0, \quad x \in \mathbb{R}^n, t > 0.$$

Applying the method of the transport equation or Fokker-Planck equation due to Carrillo-Toscani [8].
Definition. For $\alpha > 1$, we let

$$U(t,x) \equiv (1+\sigma t)^{-\frac{n}{\sigma}} \left[A - \frac{2\alpha}{\alpha - 1} \left(\frac{|x|}{(1+\sigma t)^{1/\sigma}} \right)^2 \right]_+^{1/(\alpha-1)}, \tag{5.1}$$

where $\sigma = n(\alpha-1)+2$ and A is chosen such that $\|U\|_1 = 1$.

We have the following result.

Theorem 5.1. *Let $\lambda > 0$ and $1 < \alpha < 2 - \frac{2}{n}$. Then for any positive initial data $u_0 \in L^1_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, the decaying weak solution $u(t,x)$ in Proposition 4.3 with the small initial data satisfies the following asymptotic behavior: For $M = \|u_0\|_1$,*

$$\|u(t) - MU(t)\|_1 \leq (1+\sigma t)^{-\frac{1}{\sigma} - \nu}$$

where $\sigma = n(\alpha - 1) + 2$ and $\nu = n(2-\alpha) - 2$.

Here, we only give a shortened story of the proof of Theorem 5.1. The detailed version may appear in elsewhere. Applying the method of the transport equation or Fokker-Planck equation due to Carrillo-Toscani [8], we compute the time derivative of the 2nd moment: For a weak solution u and ψ of (4.3) , we let

$$H(v(t)) \equiv \int_{\mathbb{R}^n} |x|^2 v(t) dx + \frac{2}{\alpha - 1} \int_{\mathbb{R}^n} v^\alpha(t) dx, \tag{5.2}$$

$$J(v(t)) \equiv \int_{\mathbb{R}^n} v(t) \left| \nabla \left(\frac{\alpha}{\alpha-1} v^{\alpha-1}(t) + \frac{|x|^2}{2} \right) \right|^2 dx, \tag{5.3}$$

$$I(v(t)) \equiv \int_{\mathbb{R}^n} v(t) \left| \nabla \left(\frac{\alpha}{\alpha-1} v^{\alpha-1}(t) + \frac{|x|^2}{2} - e^{-\kappa t}\phi(t) \right) \right|^2 dx. \tag{5.4}$$

It is now well understood that for a weak solution v and ϕ of (4.3) , the functional $H(v)$ plays a metric for the solution spaces: Namely we have

$$|H(v(t)) - H(v(s))| \leq 2 \int_s^t J(v(\tau)) d\tau + \int_s^t e^{-\kappa\tau} \left[\int_{\mathbb{R}^n} e^{2\tau} v^{\alpha+1}(\tau) dx + \int_{\mathbb{R}^n} |\phi(\tau)|^2 dx \right] d\tau, \tag{5.5}$$

where $\kappa = n(2-\alpha)$. In particular, for $1 < \alpha < 2 - \frac{2}{n}$, we have that $H(v(t)$ is monotonically decreasing in t and

$$H(v(t)) \leq H(u_0), \qquad t > 0. \tag{5.6}$$

The inequality (5.5) follows from the similar way we derived (4.5) in Proposition 4.1. Under the condition $1 < \alpha \le 2 - \frac{2}{n}$ we see $\kappa \ge 2$ and we have already seen that $\|v\|_{\alpha+1}^{\alpha+1} \le C$ and $\|\phi\|_2 \le C$, it follows

$$H(v(t)) \le C$$

and for some appropriate sequence $\{t_n\}_n$,

$$\begin{aligned}\lim_{n\to\infty} H(v(t_n)) - &H(u_0) + 2\int_0^\infty J(u(\tau))d\tau \\ &\le 2\int_0^\infty e^{-\kappa\tau}\int_{\mathbb{R}^n}\left(|\phi(\tau)|^2 + e^{2\tau}v^{\alpha+1}(\tau)\right)dxd\tau < \infty,\end{aligned}$$

$$\begin{aligned}&|H(v(t_n)) - H(v(t_m))| \\ \le& 2\int_{t_m}^{t_n} J(u(\tau))d\tau + 2\int_{t_m}^{t_n} e^{-\kappa\tau}\int_{\mathbb{R}^n}\left(|\phi(\tau)|^2 + e^{2\tau}v^{\alpha+1}(\tau)\right)dxd\tau \to 0, \quad n,m\to\infty\end{aligned}$$

and this shows that $\{H(v(t_n))\}_n$ is the Cauchy sequence in $n \to \infty$.

On the other hand, one may observe that the functional $I(v)$ has time decaying properties: Since $2(\kappa - 2) = 2n(2-\alpha) - 4 > 0$ under the condition $\alpha < 2 - \frac{2}{n}$, we choose η such that $\nu \equiv 2 - \eta < \min(2(\kappa-2), 1)$ and it follows that

$$I(v(t)) \le e^{-\nu t}\left(I(v_0) + C\int_0^\infty e^{\nu - 2(\kappa-2))\tau}d\tau\right) \tag{5.7}$$

This is obtained by the direct estimate for the functional $I(v)$ with aid of the regularity of the solution.

On the other hand, by a suitable subsequence t_n, $\{H(v(t_n))\}_n$ is a Cauchy sequence there exists a constant H_∞ such that $\lim_{n\to\infty} H(v(t_n)) = H_\infty$. Moreover, $d(f,g) \equiv |H(f) - H(g)|$ becomes a metric and the set

$$X = \{f \in L^\alpha(\mathbb{R}^n), xf \in L^1, f \ge 0\}$$

is a complete metric space by this metric we conclude that there exists a limit function v_∞ in X such that

$$v(t_n) \to v_\infty \quad t_n \to \infty$$

in X. While by (5.8) $I(v(t_n)) \to 0$ $(n \to \infty)$,

$$I(v_\infty) = 0$$

and we obtain $\nabla v_\infty^{\alpha-1} = -\frac{\alpha-1}{\alpha}x$. this concludes by recalling $M = \|u_0\|_1$,

$$v_\infty(x) = MV(x) = M\left[A - \frac{\alpha-1}{2\alpha}|x|^2\right]_+^{1/\alpha-1},$$

where A is chosen such that the L^1 norm of $V(x)$ is normalized as 1. Again by estimate (5.5) gives

$$0 \le H(v(t)) - H(v_\infty) \le -2\int_t^\infty I(v(\tau))d\tau + \int_t^\infty e^{-\kappa\tau}\left[\int_{\mathbb{R}^n} e^{2\tau}v^{\alpha+1}(\tau)dx + \int_{\mathbb{R}^n}|\phi(\tau)|^2dx\right]d\tau \tag{5.8}$$

the desired estimate

$$\|v(t) - MV\|_1 \le Ce^{-(\kappa-2)t}$$

follows from the Csiszar-Kulback inequality. This gives Theorem 5.1 by change of the variable into the original variables.

The entire proof relies on the regularity theorem of the degenerated parabolic equation and the crucial estimate for $I(v)$ requires some estimates. The detailed discussion will be shown in elsewhere.

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References

[1] Biler, P., *Existence and nonexistence of solutions for a model of gravitational interaction of particles, III*, Colloq. Math., **68** (1995), 229–239.

[2] Biler, P., *Local and global solvability of some parabolic systems modeling chemotaxis* Adv. Math. Sci. Appl. , **8** (1998), 715–743

[3] Biler, P., Dolbeault, J., *Long time behavior of solutions to Nernst-Planck and Debye-Hünkel drift-diffusion systems* Ann. Henry Poincaré, **1** (2000), 461–472

[4] Biler, P., Hebisch, W., Nadzieja, T., *The Debye system: existence and large time behavior of solutions* Nonlinear Anal. T.M.A., **23** (1994), 1189–1209

[5] Biler, P., Karch, G., Laurençot, Ph., Nadzieja, T., *The 8π-problem for radially symmetric solutions of a chemotaxis model in a disc.* preprint (2005)

[6] Biler, P., Nadzieja, T., Stanczy, R., *Nonisothermal systems of self-attracting Fermi-Dirac particles*, Banach Center Pulb. **66** (2004), 61–78

[7] Caffarelli, L. A., Evans, C. L., *Continuity of the temperature in two-phase Stefan problem,* Arch. Rat. Mech. Anal., **81** (1983), 199-220.

[8] Carrillo, J.A., Toscani, G., *Asymptotic L^1 decay of solutions of the porous medium equation to self- similarity*, Indiana Univ. Math. J. **49** (2000), 113-142.

[9] Dolbeault, J., Perthame, B., *Optimal critical mass in the two-dimensional Keller-Segel model in $\mathbb{R}^2$*. C.R. Math. Acad. Sci., Paris, **339** (2004), 611-616.

[10] DiBenedetto, E., Friedman, A., *Regularity of solutions of nonlinear degenerate parabolic systems*, J. reine angewandte Math. **349** (1984), 83-128.

[11] DiBenedetto, E., Friedman, A., *Hölder estimates for nonlinear degenerate paraboliec systems*, J. reine angewandte Math. **357** (1985), 1-22.

[12] Fujita, H., *On blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect.I **13** (1966), 35-37.

[13] Giga., Y., Kohn, R., *Asymptotically self-similar blow-up of semilinear heat equations.* Comm. Pure Appl. Math. **38** (1985), 297–319.

[14] Jäger, W, Luckhaus, S., *On explosions of solutions to a system of partial differential equations modeling chemotaxis,* Trans. Amer. Math. Soc., **329** (1992), 819-824.

[15] Jüngel, A., *Qualitative behavior of solutions of a degenerate nonlinear drift-diffusion model for semiconductors* Math. Model. Meth. Appl. Sci. **5** (1995), 497-518.

[16] Keller, E. F., Segel, L.A., Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., **26** (1970), 399-415.

[17] Kozono, H., Ogawa, T., Taniuchi, Y., *The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations*, Math. Z., **242** (2002), 251–278.

[18] Kozono, H., Sugiyama, Y., *Asymptotic behavior of the solution of the Keller-Segel system* preprint (2006).

[19] Kurokiba, M., Nagai, T., Ogawa, T., *The uniform boundedness and threshold for the global existence of the radial solution to a drift-diffusion system*, Comm. Pure Appl. Anal., (2006) to appear.

[20] Kurokiba M., Ogawa, T., *Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type*, Diff. Integral Equations, **16** no. 4 (2003), 427-452.

[21] Mock, M.S., *An initial value problem from semiconductor devise theory* SIAM, J. Math. **5** no 4 (1974), 597-612

[22] Nagai, T., *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci.Appl., 5 (1995), 581–601.

[23] Nagai, T., *Global existence of solutions to a parabolic system for chemotaxis in two space dimensions.* Nonlinear Anal. T.M.A., **30** (1997) 5381-5388.

[24] Nagai, T., *Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains*, J. Inequal. Appl., 6 (2001), 37–55.

[25] Nagai, T., Ogawa, T., *Brezis-Merle inequality of parabolic type and application to the global existence of the Keller-Segel equations*, preprint.

[26] Nagai, T., Senba, T., Suzuki, T., Chemotactic collapse in a parabolic system of mathematical biology, preprint

[27] Nagai, T., Senba, T., Yoshida, K., *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac., **40** (1997), no. 3, 411–433.

[28] Ogawa, T.,Taniuchi, Y., *Critical Sobolev inequality and uniqueness problem to the Navier-Stokes equations.*, Tohoku Math. J., **56** no. 1 (2004) 65-77.

[29] Ogawa, T., Tsutsumi, Y., *Blow-up of H^1 solutions for the one-dimensional nonlinear Schrödinger equation with critical power nonlinearity*, Proc. Amer. Math. Soc., **111** (1991), 487-496.

[30] Otto, F., *The geometry of Dissipative evolution equations:the porous medium equation*, Comm. P.D.E., **26** (2001), 101-174.

[31] Senba T., Suzuki, T., *Chemotaxis collapse in a parabolic-elliptic system of mathematical biology.* Adv. Differential Equations, 6 (2001), 21–50.

[32] Senba T., Suzuki, T., *Weak solutions to a parabolic elliptic system of chemotaxis,* J. Funct. Anal., 191 (2002), 17–51.

[33] Stein, E., *Singular Integrals and Differentiability Properties of Functions,*, Prinston Univ. Press, Prinston, New Jersey 1970.

[34] Sugiyama, Y., *Global existence and decay properties of solutions for some degenerate quasilinear parabolic system modeling chemotaxis* preprint.

[35] Sugiyama, Y., *Blow up in a quasilinear degenerate systems modeling chemotaxis* preprint (2005).

[36] Suzuki, T., *Method of blow up envelope and applications*, "Asymptotic Analysis and Singularities", Adv. Study Pure Math. (2006), to appear

[37] Wolansky, G., *On steady distributions of self-attracting clusters inuder friction and fluctuations*, Arch. Rat. Mech. Anal. **119** (1992), 355-391.

[38] Wolansky, G., *On the evolution of self-interacting clusters and applications to semilinear equations with exponential nonlinearity*, J. Anal. Math. **59** (1992), 251-272.

[39] Weissler, F., *Existence and non-existence of global solutions for a semilinear heat equation*, Israel J. Math., **38** (1981) 29-40.

[40] Yagi, A., *Norm behavior of solutions to a parpbolic system of chemotaxis*, Math. Japon. **45** (1997), 241–265.