L^p WELLPOSEDNESS FOR THE DRIFT-DIFFUSION SYSTEM ARISING FROM THE SEMICONDUCTOR DEVICE SIMULATION

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Abstract. We discuss a strong solution of the nonlinear parabolic system arose from the simulation for the semiconductor device design. This equation considering here is governing the electron and positive hole dynamics on the MOS FET for the Large Scaled Integral-Circuit (V-LSI). We show that the existence and uniqueness and stability of the strong solution in L^p spaces and will discuss on the global existence.

1. Introduction

Huge amount of numerical and analytical effort was executed for designing the V-LSI. The mathematical theory for the simulation for the dynamics of the electric particles has been extensively studied (see for example and [12] and references there in). Due to the demand for more fine structure on the semiconductor devises, it is now requried detailed microscopic mathematical analysis for the dynamics of the electron and holes. In the simplest model, dynamics of electron and hole density is governed by the *drift-diffsion model* which is discribed by the following coupled system of of elliptic-parabolic equations.

$$
\begin{cases}
\partial_t n - \nabla\cdot(\nu_1\nabla n - \mu_1 n\nabla\psi) = f, & t>0, x\in\mathbb{R}^n,\\
\partial_t p - \nabla\cdot(\nu_2\nabla p + \mu_2 p\nabla\psi) = f, & t>0, x\in\mathbb{R}^n,\\
-\Delta\psi = \dfrac{1}{\kappa}(p-n+g), \quad x\in\mathbb{R}^n,\\
\qquad n(0,x) = n_0(x), \qquad p(0,x) = p_0(x).
\end{cases}
\tag{1.1}
$$

where $n(t,x)$ and $p(t,x)$ denote the electron and hole density in the semiconductor, $g(x)$ denotes difference of the donor and accepter density which is given function and c ν_1, ν_2 are diffusion constants and μ_1 and μ_2 are coupled constants. $f = f(t,x)$ express the variation of the charge by the external current which is also a given function. The constant $\kappa > 0$ denotes so called the Debye length that stands for the schreening of the hole and electron particles. Mathematical study of this equation has been extensively developed,

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for example, [20], [15], [9], [8], [3], [5], [6], [2] and reference therein. The main concern of those result is devoted to the intitial boundary value problem for (**??**). Since some similar equations also appear in the other context as Nernst-Plank equation in Astromony, Keller-Siegel model in Chemotaxis, it is also interest to consider the Cauchy problem to (1.2). For simplicity, we introduce the following slightly simple system which is obtained from the above equation,

$$\begin{cases} \partial_t u - \nu\Delta u + \mu_1\nabla(u\nabla\psi) = f, & t>0, x\in\mathbb{R}^n, \\ \partial_t v - \nu\Delta v - \mu_2\nabla(v\nabla\psi) = f, & t>0, x\in\mathbb{R}^n, \\ -\Delta\psi = v-u+g, \quad x\in\mathbb{R}^n, \\ \quad u(0,x)=u_0(x), \qquad v(0,x)=v_0(x). \end{cases} \tag{1.2}$$

We first show that there exists a unique time local solution of (1.2) for any initial data in L^p.

Definition. For $1\le\theta\le\infty$, we call a pair of exponent (p,θ) as the (Serrin) admissible pair of order s if

$$\frac{2}{\theta}+\frac{n}{p}=2+s$$

and the space $L^\theta(I;L^p)$ as the Serrin class of the above scaling.

Note that if $f=0$ and $g=0$, the system of drift-diffusion holds its form under the scaling $u(t,x)\to u_\lambda(t,x)=\lambda^2 u(\lambda t^2,\lambda x)$. Under this scaling, the space $L^\theta(I;L^p)$ is invariant for the 0 order admissible pair (p,θ). Hence it is expectable that there is a local well-posedness result in the space of 0 order Serrin class.

Theorem 1.1. *Suppose that*

$$\begin{aligned} &1<p<2, && \text{for } n=2 \\ &\frac{n}{2}\le p<n && \text{for } n\ge 3 \end{aligned} \tag{1.3}$$

and $1\le\theta<2$, *let* $f\in L^\theta(0,T;W^{1,p})$ *and* $g\in W^{1,p}$. *Then for* $(u_0,v_0)\in L^p\times L^p$, *there exists* $T>0$ *and a unique solution of (1.2) with the initial data* (u_0,v_0) *such that* u, $v\in C([0,T);L^p)\cap C((0,T);W^{2,p})\cap C^1((0,T);L^p)$. *When* $n=2$ *and* $p=\theta=2$, *the same result also holds.*

For the case of lower exponent of p, the regularity for the initial data is required for the well posedness.

Theorem 1.2. *For* $0<a\le 1$, *we let* $\frac{2n}{n+a+1}<p<\frac{n}{a+1}$, $2a<\theta<\frac{2}{1-\sigma}$. *Suppose that* $f\in L^\theta(0,T;W^{1,p})$ *and* $g\in W^{1,p}$. *Then for* $(u_0,v_0)\in W^{\sigma,p}\times W^{\sigma,p}$, *there exists* $T>0$ *and unique solution of (1.2) with the initial data* (u_0,v_0) *such that* u, $v\in C([0,T);L^p)\cap C((0,T);W^{2,p})\cap C^1((0,T);L^p)$.

Remark. There are many results for the existence and uniqness problem for the initial boundary value problem corresponding to (1.2) in a bounded domain $\Omega\subset\mathbb{R}^3$ with the

Neumann and other boundary conditions. See for example, [15], [3], [9], [8] and [12]. Also some abstract resuls can also covers some cases. See for this direction, [10], [11], [23]. Here the difference with our result than the previous one is that the domain is not bounded and the initial data is rather weaker. That is we assume the less regularity for the initial data. This is related to the initial behavior for the solution near $t = 0$.

Under more restricted situataion, we can state the global existence for the system (1.2).

Theorem 1.3. *Supporse that $g \in L^\infty(\mathbb{R}^n)$, $f \equiv 0$ and we assume that $1 < p \le 2$ with $\theta < 2$. Supporse that the initial data satisfies the same assupntion as in Theorem 1.1 and moreover they are positive definite, i.e, $n_0(x), p_0(x) \ge 0$. Then the solution obtained in Theorem 1.1 exists globally in time i.e., (n,p) in $C([0,T); W^{2,p}) \cap C^1((0,\infty); L^2)$.*

If $p = 2$, the equation is globally wellposed in $C([0,\infty); W^{2,2}) \cap C^1((0,\infty); L^2)$.

We should also mention that there is a very much related research for a similar system. First, the drift-diffusion system equation has a strong analogy to the equation of the fluid mechanics. From the Navier-Stokes equtaion; the incompressible fluid dynamics is described by the well-known system:

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = f, \qquad t > 0, x \in \mathbb{R}^n, \\ \quad \text{div } u = 0, \qquad t > 0, x \in \mathbb{R}^n, \\ \qquad u(0,x) = u_0(x), \end{cases} \tag{1.4}$$

one can reduce the equation to the vorticity; By setting $\omega = \text{rot } u$ we have

$$\begin{cases} \partial_t \omega - \nu\Delta\omega + u\cdot\nabla\omega - \omega\cdot\nabla u = \text{rot } f, \qquad t > 0, x \in \mathbb{R}^n, \\ \quad \text{div } \omega = 0, \qquad t > 0, x \in \mathbb{R}^n, \\ \qquad \omega(0,x) = \text{rot } u_0(x), \end{cases} \tag{1.5}$$

Obsearving that the velocity u can be inverted by the vorticity by the Bio-Savourt law as

$$u = \text{rot } (-\Delta)^{-1}\omega.$$

we see the drift-diffusion system (1.2) has an analogous stracture of nonlinearity to the vorticity equation (1.5).

The other one is for the model of chemotaxtics. The dynamics of the dencity of mucus is govorned by the following system of parabolic equations called as the Keller-Segel system.

$$\begin{cases} \partial_t u - \nu\Delta u + \chi\nabla(u\nabla\psi) = 0, \qquad t > 0, x \in \Omega, \\ \mu\partial_t\psi - \nu\Delta\psi + \gamma\psi = \alpha u, \qquad t > 0, x \in \Omega, \\ \qquad u(0,x) = u_0(x), \qquad \psi(0,x) = \psi_0(x), \\ \qquad \dfrac{\partial u}{\partial n} = \dfrac{\partial\psi}{\partial n} = 0 \qquad x \in \partial\Omega, \end{cases} \tag{1.6}$$

where $\mu, \nu, \chi, \gamma, \alpha$ are positive constants. When the parameter $\mu = 0$, the system is elliptic-parabolic and very close looking to the drift-diffusion model. However the apperence of positive constant $\gamma > 0$ makes the L^1 local wellposedness for (1.6) rather easier since it does not involve the singular integral operator. On the other hand for the global existnce, it is observed that the solution blows-up in a finite time in the case $\mu = 1$. Our system (1.2) is however more stable and there is no blow up solution like that in (1.6).

To solve those equations, we first invert them into the corresponding integral equation via the Duhamel principle. Namely we find a solution to

$$\tag{1.7} \begin{cases} u = e^{\nu t\Delta}u_0 - \displaystyle\int_0^t e^{\nu(t-t')\Delta}\{\mu_1\nabla(u(\tau)\nabla\psi(\tau)) - f(\tau)\}d\tau, \qquad t > 0, \\ v = e^{\nu t\Delta}u_0 + \displaystyle\int_0^t e^{\nu(t-t')\Delta}\{\mu_2\nabla(v(\tau)\nabla\psi(\tau)) + f(\tau)\}d\tau, \\ -\Delta\psi = v - u + g, \quad x \in \mathbb{R}^n. \end{cases}$$

Since $-\Delta$ is invertible in $\mathbb{R}^n$ by the Newton potential

$$\tag{1.8} \psi = (-\Delta)^{-1}(v-u+g) = \begin{cases} \dfrac{1}{n(n-2)\omega_n}\displaystyle\int_{\mathbb{R}^n}\frac{v(y)-u(y)+g(y)}{|x-y|^{n-2}}dy, \quad n \ge 3 \\ \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^n}(v(y)-u(y)+g(y))\log|x-y|^{-1}dy, \quad n = 2, \end{cases}$$

where ω_n denotes the surface are of the unit sphere, we can eliminate ψ from the integral equation and obtain

(1.9)

$$\begin{cases} u = e^{\nu t\Delta}u_0 - \displaystyle\int_0^t e^{\nu(t-t')\Delta}\left\{\mu_1\nabla(u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)) - f(\tau)\right\}d\tau, \qquad t > 0, \\ v = e^{\nu t\Delta}u_0 + \displaystyle\int_0^t e^{\nu(t-t')\Delta}\left\{\mu_2\nabla(u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)) + f(\tau)\right\}d\tau. \end{cases}$$

The solution can be constracted by the fixed point argument in the properly defined function space where the initial data only assumed in $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$.

2. Preliminary Lemma

We summarize some lemmas that will be used in the proof of the existence of solutions. The first one is the linear estimate for the heat kernel which is originally due to Serrin, Weissler and Giga.

Proposition 2.1 (Serrin, Weissler, Giga). *For $\nu > 0$, $1 \le q \le p \le \infty$ $\gamma = n/2(1/q - 1/p)$,*

(1) *For $\phi \in L^q$ and $0 \le \alpha < 1$,*

$$\|\nabla^\alpha e^{\nu t\Delta}\phi\|_p \le C_0|t|^{-\gamma-\alpha/2}\|\phi\|_q.$$

Next we states some well-known general inequalities.

Proposition 2.2 (Hardy-Littlewood-Sobolev). *Let $f \in L^p$ and $I_\mu f = c_n|x|^{-(n-\mu)} * f$ where $0 < \mu < n$. Then we have*

$$\|I_\mu f\|_q \le C_{n,\mu}\|f\|_p \qquad \frac{1}{q} = \frac{1}{p} - \frac{\mu}{n}.$$

For the proof, see Ziemer [24].

Proposition 2.2 immediately gives

Corollary 2.3. *Let $f \in L^p$ and $(-\Delta)^{-1} f = C_n I_2 f$ where $n \geq 2$. Then we have*

$$\|\nabla(-\Delta)^{-1} f\|_q \leq C_n \|f\|_p \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

3. Proof of Existence and Regularity

We begin for the proof of Theorem 1.1. For the given initial data $(u_0, v_0) \in L^p \times L^p$, we choose $M = C_1\|u_0\|_p \vee C_1\|f\|_{L^\theta(I;L^p)}$, $N = C_1\|v_0\|_p \vee C_1\|f\|_{L^\theta(I;L^p)}$ and $G = \|g\|_p$. The solution can be constructed in the complete metric space

$$X_T = \left\{(\phi, \psi) \in \left\{C([0,T); L^p) \cap L^\theta(0,T; \dot{W}^{1,p})\right\}^2; \|\!|\phi\|\!|_X \leq 3M, \|\!|\psi\|\!|_X \leq 3N\right\},$$

where

$$\|\!|\phi\|\!|_X \equiv \sup_{t\in[0,T)} \|\phi\|_p + \left(\int_0^T \|\nabla\phi\|_p^\theta d\tau\right)^{1/\theta}.$$

Let

(3.1)
$$\begin{cases} \Phi(u,v)(t) \equiv e^{\nu t\Delta} u_0 - \displaystyle\int_0^t e^{\nu(t-t')\Delta}\{\mu_1 \nabla(u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)) - f(\tau)\}d\tau, \\ \Psi(u,v)(t) \equiv e^{\nu t\Delta} v_0 + \displaystyle\int_0^t e^{\nu(t-t')\Delta}\{\mu_2 \nabla(v(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)) + f(\tau)\}d\tau. \end{cases}$$

The following estimate is essential for proving the existence.

Proposition 3.1. *Let $(u_0, v_0) \in (L^p)^2$. Then for small $T > 0$ and $(u,v) \in X_T$, we have*

$$\|\!|\Phi(u,v)\|\!|_X \leq 3M \tag{3.2}$$

$$\|\!|\Psi(u,v)\|\!|_X \leq 3N \tag{3.3}$$

$$\|\!|\Phi(u_1,v_1) - \Phi(u_2,v_2)\|\!|_X \leq \frac{1}{4}\{\|\!|u_1-u_2\|\!|_X + \|\!|v_1-v_2\|\!|_X\} \tag{3.4}$$

$$\|\!|\Psi(u_1,v_1) - \Psi(u_2,v_2)\|\!|_X \leq \frac{1}{4}\{\|\!|u_1-u_2\|\!|_X + \|\!|v_1-v_2\|\!|_X\} \tag{3.5}$$

Proof of Proposition 3.1. We first show (3.2) -(3.5) under the condition

$$\frac{n}{2} < p < n, \qquad \frac{2n}{n+1} \leq p < n.$$

By Proposition 2.1 and Corollary 2.3

(3.6)

$$
\begin{aligned}
\|\Phi(u,v)\|_p \leq & \|u_0\|_p + C\mu_1 \int_0^t (t-\tau)^{-\gamma-1/2} \left\|u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)\right\|_r d\tau \\
& + C\int_0^t (t-\tau)^{-\gamma}\|f(\tau)\|_r d\tau \\
\leq & M + C\mu_1 \int_0^t (t-\tau)^{-\gamma-1/2}\|u(\tau)\|_p\|\nabla(-\Delta)^{-1}(v-u+g)(\tau)\|_q d\tau \\
& + C\int_0^t (t-\tau)^{-\gamma}\|f(\tau)\|_r d\tau \\
\leq & M + C\mu_1 \int_0^t (t-\tau)^{-\gamma}\|u(\tau)\|_p\||(v-u+g)(\tau)\|_p d\tau \\
& + C\int_0^t (t-\tau)^{-\gamma}\|f(\tau)\|_r d\tau,
\end{aligned}
$$

where

$$
\gamma = \frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right) \in [0,1/2)\,, \qquad \frac{1}{r} = \frac{1}{p}+\frac{1}{q}, \qquad \frac{1}{q} = \frac{1}{p}-\frac{1}{n} \in (0,1],
$$

and it is required that $1/r \in (0,1]$ and $\gamma \in [0,1)$, $1/q \in (0,1)$. This can be satisfied under the condition

(3.7)
$$
\begin{aligned}
&\frac{n}{2} < p < n \\
&\frac{2n}{n+2} \leq p < n
\end{aligned}
$$

Thus using Proposition 2.2 for $I = [0,T)$,

(3.8)

$$
\begin{aligned}
\|\Phi(u,v)\|_{L^\infty(I;L^p)} \leq & M + C\mu_1 \sup_{\tau\in I}\|u(\tau)\|_p \left\| |t|^{-\gamma-1/2} * \|(v-u+g)(\tau)\|_p \right\|_{L^\infty(I)} \\
& + \left\| |\tau|^{-\gamma} * f(\tau)\|_r \right\|_{L^\infty(I)} \\
\leq & M + C\mu_1 \sup_{\tau\in I}\|u(\tau)\|_p(\|v(t)\|_{L^\rho(I;L^p)} + \|u(\tau)\|_{L^\rho(I;L^p)} + \|g\|_{L^\rho(I;L^p)}) \\
& + C\|f(\tau)\|_{L^{\frac{1}{1-\gamma}}(I;L^r)} \\
\leq & M + C\mu_1 T^{1/\rho}\sup_{\tau\in I}\|u(\tau)\|_p(\sup_{\tau\in I}\|v(\tau)\|_p + \sup_{\tau\in I}\|u(\tau)\|_p + \|g\|_p) \\
& + CT^\beta\|f(\tau)\|_{L^\theta(I;L^r)} \\
\leq & M + C(\mu_1 T^{1/\rho} + T^{1/\theta})3M(3M+3N+G),
\end{aligned}
$$

where

$$
\frac{1}{\rho} = \frac{1}{2} - \gamma \equiv \delta > 0 \qquad \beta = \frac{1}{\theta} - 1 + \gamma.
$$

The last condition requires $\frac{n}{2} < p$ and $\theta < 2$. Hence by choosing T properly small, we have

$$\|\Phi(u,v)\|_{L^\infty(I;L^p)} \le \frac{3}{2}M. \tag{3.9}$$

Similarly for $\gamma = \frac{n}{2}(1/r - 1/p)$,

$$\begin{aligned}
\|\nabla\Phi(u,v)\|_p \le & C_0 t^{-1/2}\|u_0\|_p \\
& + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}} \|\nabla\{u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)\}\|_r d\tau \\
& + C\int_0^t (t-\tau)^{-\gamma-1/2}\|f(\tau)\|_r d\tau \\
\le & C_0 t^{-1/2}\|u_0\|_p + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}} \big\{\|\nabla u(\tau)\|_p \|\nabla(-\Delta)^{-1}(v-u+g)(\tau)\|_q \\
& \qquad + \|u(\tau)\|_q \|\nabla^2(-\Delta)^{-1}(v-u+g)(\tau)\|_p\big\} d\tau \\
& + C\int_0^t (t-\tau)^{-\gamma-\alpha/2}\|f(\tau)\|_r d\tau
\end{aligned}$$

(by Corollary 2.3 and Sobolev's inequatlity)

$$\begin{aligned}
\le & C_0 t^{-1/2}\|u_0\|_p + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}} \|\nabla u(\tau)\|_p \|(v-u+g)(\tau)\|_p d\tau \\
& + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}} \|u(\tau)\|_q \|(v-u+g)(\tau)\|_p d\tau \\
& + C\int_0^t (t-\tau)^{-\gamma-\alpha/2}\|f(\tau)\|_r d\tau,
\end{aligned}$$

(by the Sobolev inequality)

$$\begin{aligned}
\le & C_0 t^{-1/2}\|u_0\|_p + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}} \|\nabla u(\tau)\|_p \|(v-u+g)(\tau)\|_p d\tau \\
& + C\int_0^t (t-\tau)^{-\gamma-\alpha/2}\|f(\tau)\|_r d\tau,
\end{aligned}$$

where

$$\gamma = \frac{n}{2}(\frac{1}{r} - \frac{1}{p}), \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

Thus we have for $I = [0, T)$ and $\theta \in [1, 2)$,

(3.10)

$$\begin{aligned}
\|\nabla\Phi(u, v)\|_{L^\theta(I;L^p)} &\le M + C\mu_1 \sup_{\tau\in I}\{\|u(\tau)\|_p + \|v(\tau)\|_p + \|g\|_p\} \left\| |t|^{-\gamma-1/2} * \|\nabla u(\tau)\|_p \right\|_{L^\theta(I)} \\
&\quad + C \left\| |\tau|^{-\gamma-1/2} * \|f(\tau)\|_r \right\|_{L^\theta(I)} \\
&\le M + C\mu_1\{\|u\|_{L^\infty(I;L^p)} + \|v\|_{L^\infty(I;L^p)} + \|g\|_p\}\|\nabla u\|_{L^\rho(I;L^p)} \\
&\quad + C\|f\|_{L^\rho(I;L^r)} \\
&\le M + C\mu_1 T^{1/\rho-1/\theta}\{\|u(\tau)\|_{L^\infty(I;L^p)} + \|v(\tau)\|_{L^\infty(I;L^p)} + \|g\|_p\}\|\nabla u\|_{L^\theta(I;L^p)} \\
&\quad + CT^{1/\rho-1/.\theta}\|f(\tau)\|_{L^\theta(I;L^r)} \\
&\le M + 9C(\mu_1 + 1)T^\delta M(M + N + G) + C\mu_1(T^{1/\rho} + T^\delta)M,
\end{aligned}$$

where

$$\frac{1}{r} = \frac{2}{p} - \frac{1}{n}, \qquad \frac{1}{\rho} - \frac{1}{\theta} = \frac{1}{2} - \gamma = 1 - \frac{n}{2p} \equiv \delta > 0.$$

Hence again by choosing T sufficiently small, we conclude

$$\|\nabla^\alpha\Phi(u, v)\|_{L^\theta(I;L^p)} \le \frac{3}{2}M. \tag{3.11}$$

Inequalities (4.4) and (4.6) imply (3.2). The inequality (3.3) follows in a simlar way.

Next we consider the third inequality (3.4). Let (u_1, v_1) and $(u_2, v_2) \in X_T$.

(3.12)

$$\begin{aligned}
\|\Phi(u_1, v_1) - \Phi(u_2, v_2)\|_p \le& C\mu_1 \int_0^t \|\nabla e^{\nu t\Delta}(u_1(\tau) - u_2(\tau))\nabla(-\Delta)^{-1}(v_1 - u_1 + g)(\tau))\|_r d\tau \\
&+ C\mu_1 \int_0^t \|\nabla e^{\nu t\Delta} u_2(\tau)\nabla(-\Delta)^{-1}(v_1 - v_2 - u_1 + u_2)(\tau)\|_r d\tau \\
\le& C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}}\|u_1(\tau) - u_2(\tau)\|_p\|\nabla(-\Delta)^{-1}(v_1 - u_1 + g)(\tau)\|_q d\tau \\
&+ C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}}\|u_2(\tau)\|_p\|\nabla(-\Delta)^{-1}(v_1 - v_2)\|_q d\tau \\
&+ C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}}\|u_2(\tau)\|_p\|\nabla(-\Delta)^{-1}(u_1 - u_2)\|_q) d\tau \\
\le& C\mu_1 T^{1/\rho} \sup_{\tau\in I}\|u_1(\tau) - u_2(\tau)\|_p \sup_{\tau\in I}(\|(v_1(\tau)\| + \|u_1(\tau)\|_p + \|g\|_p) \\
&+ C\mu_1 T^{1/\rho} \sup_{\tau\in I}\|u_2(\tau)\|_p(\sup_{\tau\in I}\|v_1 - v_2\|_p + \sup_{\tau\in I}\|u_1 - u_2\|_p) \\
\le& C\mu_1 T^{1/\rho}\{(M + N + G)\|u_1 - u_2\|_X + M\|v_1 - v_2\|_X\}
\end{aligned}$$

where

$$\gamma = \frac{n}{2}(\frac{1}{r} - \frac{1}{p}), \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

(3.13)
$$\begin{aligned}\|\nabla\Phi(u_1,v_1)-&\nabla\Phi(u_2,v_2)\|_p\\ \le& C\mu_1\int_0^t(t-\tau)^{-\gamma-\frac{1}{2}}\big\{\|\nabla(u_1(\tau)-u_2(\tau))\|_p\|\nabla(-\Delta)^{-1}(v_1-u_1+g)(\tau)\|_q\\ &+\|u_2(\tau)\|_q\|\nabla^2(-\Delta)^{-1}(v_1-v_2-u_1+u_2)(\tau)\|_p\big\}d\tau.\end{aligned}$$

Similarly to (4.5)

(3.14)
$$\begin{aligned}\|\nabla(\Phi(u_1,v_1)-&\Phi(u_2,v_2))\|_{L^\theta(I;L^p)}\\ \le& C\mu_1\sup_{\tau\in I}\{\|u_1(\tau)\|_p+\|v_1(\tau)\|_p+\|g\|_p\}\|\nabla(u_1-u_2)\|_{L^\rho(I;L^p)}\\ &+C\mu_1\{\|\nabla(u_1-u_2)\|_{L^\rho(I;L^p)}+\|\nabla(v_1-v_2)\|_{L^\rho(I;L^p)}\}\|u_2\|_{L^\infty(I;L^p)}\\ \le& C\mu_1T^\delta(3M+3N+G)\|\nabla(u_1-u_2)\|_{L^\theta(I;L^p)}\\ &+3C\mu_1T^\delta M\left\{\|\nabla(u_1-u_2)\|_{L^\rho(I;L^p)}+\|\nabla(v_1-v_2)\|_{L^\rho(I;L^p)}\right\}\\ \le& 3C(\mu_1+1)T^\delta(M+N+G)(\|u_1-u_2\|_X+\|v_1-v_2\|_X).\end{aligned}$$

Hence (3.12) and (3.14) imply (3.4). (3.5) is similary obtained.

□

Proof of Theorem 1.1, continued. Since Propsotion 3.1 shows that for properly small $T>0$,

$$(\Phi(u,v),\Psi(u,v)):X_T\to X_T$$

is a contraction mapping. By the Banach fixed point theorem, there exists a unique fixed point $(u,v)\in X_T$ such that

(3.15)
$$\begin{aligned}u(t)&=\Phi(u,v)(t)=e^{\nu t\Delta}u_0-\int_0^t e^{\nu(t-t')\Delta}\{\mu_1\nabla(u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau))-f(\tau)\}d\tau,\\ v(t)&=\Psi(u,v)(t)\equiv e^{\nu t\Delta}v_0+\int_0^t e^{\nu(t-t')\Delta}\{\mu_2\nabla(v(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau))+f(\tau)\}d\tau\end{aligned}$$

hold.

Now for any small $0<\eta<T$, $(u(\eta),v(\eta))\in W^{1,p}$ and we exchange the initial data such as $(u_1,v_1)=(u(\eta),v(\eta))$. Then the regularity of solution follows in the similar argument of the above.

For the given initial data $(u_0,v_0)\in W^{1,p}\times W^{1,p}$, we choose $M=C_1\|u_1\|_p\vee C_1\|f\|_{L^\theta(I;W^{1,p})}$, $N=C_1\|v_1\|_p\vee C_1\|f\|_{L^\theta(I;W^{1,p})}$ and $G=\|g\|_{W^{1,p}}$, where $I=[0,T']$. The solution can be constructed in the complete metric space

$$Y_T=\left\{(\phi,\psi)\in\left\{C([0,T');W^{1,p})\cap L^\theta(0,T';\dot W^{2,p})\right\}^2;\|\phi\|_Y\le 3M,\|\psi\|_Y\le 3N\right\},$$

where

$$\|\phi\|_Y \equiv \sup_{t\in[0,T)} \|\phi\|_{W^{1,p}} + \left(\int_0^T \|\Delta\phi\|_p^\theta d\tau\right)^{1/\theta}.$$

Let

(3.16)
$$\begin{cases} \Phi(u,v)(t) \equiv e^{\nu t\Delta}u_1 - \displaystyle\int_0^t e^{\nu(t-t')\Delta}\{\mu_1\nabla(u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)) - f(\tau)\}d\tau, \\ \Psi(u,v)(t) \equiv e^{\nu t\Delta}v_1 + \displaystyle\int_0^t e^{\nu(t-t')\Delta}\{\mu_2\nabla(v(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)) + f(\tau)\}d\tau. \end{cases}$$

The following estimate is essential for proving the existence.

Proposition 3.2. *Let $(u_1, v_1) \in (W^{1,p})^2$. Then for small $T' > 0$ with $T' < T$ and $(u,v) \in Y_{T'}$, we have*

$$\|\Phi(u,v)\|_Y \le 3M \tag{3.17}$$
$$\|\Psi(u,v)\|_Y \le 3N \tag{3.18}$$
$$\|\Phi(u_1,v_1) - \Phi(u_2,v_2)\|_Y \le \frac{1}{4}\{\|u_1-u_2\|_Y + \|v_1-v_2\|_Y\} \tag{3.19}$$
$$\|\Psi(u_1,v_1) - \Psi(u_2,v_2)\|_Y \le \frac{1}{4}\{\|u_1-u_2\|_Y + \|v_1-v_2\|_Y\} \tag{3.20}$$

Proof of Proposition 3.2. The proof is very much similar to the one of Proposition 3.1. For example,

(3.21)
$$\begin{aligned} \|\Phi(u,v)\|_{W^{1,p}} \le& \|u_0\|_{W^{1,p}} + C\mu_1\int_0^t (t-\tau)^{-\gamma-\frac{1}{2}}\|u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)\|_{W^{1,r}}d\tau \\ &+ C\int_0^t (t-\tau)^{-\gamma}\|f(\tau)\|_{W^{1,r}}d\tau \\ \le& M + C\mu_1\int_0^t (t-\tau)^{-\gamma-\frac{1}{2}}\|u(\tau)\|_{W^{1,p}}\|(v-u+g)(\tau)\|_{W^{1,p}}d\tau \\ &+ C\int_0^t (t-\tau)^{-\gamma}\|f(\tau)\|_{W^{1,r}}d\tau, \end{aligned}$$

where

$$\gamma = \frac{n}{2}(\frac{1}{r} - \frac{1}{p}), \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

Thus for $I = [0, T')$,

(3.22)

$$\begin{aligned}
\|\Phi(u,v)\|_{L^\infty(I;W^{1,p})} \le & M + C\mu_1 T^{1/\rho} \sup_{\tau\in I} \|u(\tau)\|_{W^{1,p}} \\
& \times (\sup_{\tau\in I} \|v(\tau)\|_{W^{1,p}} + \sup_{\tau\in I} \|u(\tau)\|_{W^{1,p}} + \|g\|_{W^{1,p}}) \\
& + CT^\beta \|f(\tau)\|_{L^\theta(I;W^{1,r})} \\
\le & M + C(\mu_1 T^{1/\rho} + T^{1/\theta}) 3M(3M + 3N + G),
\end{aligned}$$

where

$$\frac{1}{\rho} = \frac{1}{2} - \gamma \qquad \beta = \frac{1}{\theta} - 1 + \gamma.$$

The last condition requires $\theta < 2$.

Similarly for $\gamma = \frac{n}{2}(1/r - 1/p)$,

(3.23)

$$\begin{aligned}
\|\Delta\Phi(u,v)\|_p \le & C_0 t^{-1/2} \|u_0\|_{W^{1,p}} \\
& + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}} \|\nabla\{u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)\}\|_{W^{1,r}} d\tau \\
& + C\int_0^t (t-\tau)^{-\gamma-1/2} \|f(\tau)\|_{W^{1,r}} d\tau \\
\le & C_0 t^{-1/2} \|u_0\|_{W^{1,p}} + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}} \|\nabla u(\tau)\|_p \|(v-u+g)(\tau)\|_{W^{1,p}} d\tau \\
& + C\int_0^t (t-\tau)^{-\gamma-\alpha/2} \|f(\tau)\|_{W^{1,r}} d\tau,
\end{aligned}$$

where

$$\gamma = \frac{n}{2}(\frac{1}{r} - \frac{1}{p}), \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

Hence we obtain for small T' that

$$|\!|\!|\Phi(u,v)|\!|\!|_Y \le 3M.$$

The estimate (3.18) follows similarly. For the difference estimates (3.19) and (3.20), the proof is also varid for the esimates (3.12) to (3.14) changing $\|\cdot\|_p$ into $\|\cdot\|_{W^{1,p}}$. ❑

Proof of Theorem 1.1, concluded. By Proposition 3.2, one can show the existence of the solution of (3.16) with the initial data (u_1, v_1) which has a better regularity over (η, T) than the one obtained in X_T. Namely it is constracted a smooth pair of solutions $(u,v) \in C([\eta, T'); W^{1,p}) \cap L^\theta((\eta, T'); W^{2,p})$. This solution coincides of the original solution by the uniquness of the solution of (3.16) in X_T. We then exchange the initial data and obtain the regularity upto the time interval $(0, T)$.

After reiterating these procedure, we obtain a regular solution (u, v). It is then standard to show that the solution of (3.16) satisfies the equation (1.2).

4. Proof of Existence and Regularity

Proof of Theorem 1.2. Let $0 < \sigma < 1$. For the given initial data $(u_0, v_0) \in W^{\sigma,p} \times W^{\sigma,p}$, we change $M = C_1\|u_0\|_{W^{\sigma,p}} \vee C_1\|f\|_{L^\theta(I;L^p)}$, $N = C_1\|v_0\|_{W^{\sigma,p}} \vee C_1\|f\|_{L^\theta(I;L^p)}$ and $G = \|g\|_p$ and we also modify the complete metric space as

$$X_T^\sigma = \left\{(\phi,\psi) \in \left\{C([0,T);W^{\sigma,p}) \cap L^\theta(0,T;\dot{W}^{1,p})\right\}^2; \|\phi\|_X \le 3M, \|\psi\|_X \le 3N\right\},$$

where

$$\|\phi\|_X \equiv \sup_{t\in[0,T)} \|\phi\|_{W^{\sigma,p}} + \left(\int_0^T \|\nabla\phi\|_p^\theta d\tau\right)^{1/\theta}.$$

❑

By changing this norm, the similar estimate in Proposition3.1 can be obtained. In fact, By Proposition 2.1 and Corollary 2.3 and Gagliard-Nirenberg inequality.

(4.1)

$$\begin{aligned}
\|\Phi(u,v)\|_p \le& \|u_0\|_p + C\mu_1 \int_0^t (t-\tau)^{-\gamma-1/2} \left\|u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)\right\|_r d\tau \\
&+ C\int_0^t (t-\tau)^{-\gamma}\|f(\tau)\|_r d\tau \\
\le& M + C\mu_1 \int_0^t (t-\tau)^{-\gamma-1/2}\|u(\tau)\|_p \|\nabla(-\Delta)^{-1}(v-u+g)(\tau)\|_q d\tau \\
&+ C\int_0^t (t-\tau)^{-\gamma}\|f(\tau)\|_r d\tau \\
\le& M + C\mu_1 \int_0^t (t-\tau)^{-\gamma}\|u(\tau)\|_p^{2-a} \|(v-u+g)(\tau)\|_p^a d\tau \\
&+ C\int_0^t (t-\tau)^{-\gamma}\|f(\tau)\|_r d\tau, \\
\le& M + C\mu_1 \int_0^t (t-\tau)^{-\gamma}\|u(\tau)\|_p^2 + \|(v-u+g)(\tau)\|_p^{2-a} d\tau \\
&+ C\int_0^t (t-\tau)^{-\gamma}\|f(\tau)\|_r d\tau,
\end{aligned}$$

where

$$\gamma = \frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right) \in [0,1/2), \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \qquad \frac{1}{q} = \frac{1}{q'} - \frac{1+\alpha}{n} \in (0,1],$$
$$0 < a < 1, \frac{1}{q'} = \frac{1}{p} + \frac{\alpha}{2} - \frac{a}{2} \in (0,1], \quad \alpha \le a \le 1.$$

and it is required that $1/r \in (0,1]$ and $\gamma \in [0,1)$, $1/q \in (0,1)$. This can be satisfied under the condition

$$0 < a < 1$$
$$\frac{2n}{n+a+1} \le p < \frac{n}{a+1} \tag{4.2}$$

Thus using Proposition 2.2 for $I = [0, T)$,

(4.3)

$$\begin{aligned}
\|\Phi(u,v)\|_{L^\infty(I;L^p)} \le& M + C\mu_1 \sup_{\tau\in I} \|u(\tau)\|_p \left\| |t|^{-\gamma-1/2} * \|(v-u+g)(\tau)\|_p \right\|_{L^\infty(I)} \\
&+ \left\| |\tau|^{-\gamma} * f(\tau)\|_r \right\|_{L^\infty(I)} \\
\le& M + C\mu_1 \sup_{\tau\in I} \|u(\tau)\|_p (\|v(t)\|_{L^\rho(I;L^p)} + \|u(\tau)\|_{L^\rho(I;L^p)} + \|g\|_{L^\rho(I;L^p)}) \\
&+ C\|f(\tau)\|_{L^{\frac{1}{1-\gamma}}(I;L^r)} \\
\le& M + C\mu_1 T^{1/\rho} \sup_{\tau\in I} \|u(\tau)\|_p (\sup_{\tau\in I} \|v(\tau)\|_p + \sup_{\tau\in I} \|u(\tau)\|_p + \|g\|_p) \\
&+ CT^\beta \|f(\tau)\|_{L^\theta(I;L^r)} \\
\le& M + C(\mu_1 T^{1/\rho} + T^{1/\theta}) 3M(3M + 3N + G),
\end{aligned}$$

where

$$\frac{1}{\rho} = \frac{1}{2} - \gamma \equiv \delta > 0 \qquad \beta = \frac{1}{\theta} - 1 + \gamma.$$

The last condition requires $\frac{n}{2} < p$ and $\theta < 2$. Hence by choosing T properly small, we have

$$\|\Phi(u,v)\|_{L^\infty(I;L^p)} \le \frac{3}{2} M. \tag{4.4}$$

Similarly for $\gamma = \frac{n}{2}(1/r - 1/p)$,

$$\begin{aligned}
\|\nabla\Phi(u,v)\|_p \le& C_0 t^{-(1-\sigma)/2} \|\nabla^\sigma u_0\|_p \\
&+ C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}} \|\nabla\{u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)\}\|_r d\tau \\
&+ C\int_0^t (t-\tau)^{-\gamma-1/2} \|f(\tau)\|_r d\tau \\
\le& C_0 t^{-(1-\sigma)/2} \|u_0\|_p + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}} \big\{ \|\nabla u(\tau)\|_p \|\nabla(-\Delta)^{-1}(v-u+g)(\tau)\|_q \\
&+ \|u(\tau)\|_q \|\nabla^2(-\Delta)^{-1}(v-u+g)(\tau)\|_p \big\} d\tau \\
&+ C\int_0^t (t-\tau)^{-\gamma-\alpha/2} \|f(\tau)\|_r d\tau
\end{aligned}$$

(by Corollary 2.3 and Sobolev's inequatlity)

$$
\begin{aligned}
\leq & C_0 t^{-(1-\sigma)/2}\|u_0\|_p + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}}\|\nabla u(\tau)\|_p\|(v-u+g)(\tau)\|_p d\tau \\
& + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}}\|u(\tau)\|_q\|(v-u+g)(\tau)\|_p d\tau \\
& + C\int_0^t (t-\tau)^{-\gamma-\alpha/2}\|f(\tau)\|_r d\tau,
\end{aligned}
$$

(by the Sobolev inequality)

$$
\begin{aligned}
\leq & C_0 t^{-1/2}\|u_0\|_p + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}}\|\nabla u(\tau)\|_p\|(v-u+g)(\tau)\|_p d\tau \\
& + C\int_0^t (t-\tau)^{-\gamma-\alpha/2}\|f(\tau)\|_r d\tau,
\end{aligned}
$$

where

$$
\gamma = \frac{n}{2}(\frac{1}{r}-\frac{1}{p}), \qquad \frac{1}{r} = \frac{1}{p}+\frac{1}{q} \qquad \frac{1}{q} = \frac{1}{p}-\frac{1}{n}.
$$

Thus we have for $I=[0,T)$ and $\theta\in[1,2)$,

(4.5)

$$
\begin{aligned}
\|\nabla\Phi(u,v)\|_{L^\theta(I;L^p)} \leq & M + C\mu_1 \sup_{\tau\in I}\{\|u(\tau)\|_p + \|v(\tau)\|_p + \|g\|_p\} \left\| |t|^{-\gamma-1/2} * \|\nabla u(\tau)\|_p \right\|_{L^\theta(I)} \\
& + C\left\| |\tau|^{-\gamma-1/2} * \|f(\tau)\|_r \right\|_{L^\theta(I)} \\
\leq & M + C\mu_1\{\|u\|_{L^\infty(I;L^p)} + \|v\|_{L^\infty(I;L^p)} + \|g\|_p\}\|\nabla u\|_{L^\rho(I;L^p)} \\
& + C\|f\|_{L^\rho(I;L^r)} \\
\leq & M + C\mu_1 T^{1/\rho-1/\theta}\{\|u(\tau)\|_{L^\infty(I;L^p)} + \|v(\tau)\|_{L^\infty(I;L^p)} + \|g\|_p\}\|\nabla u\|_{L^\theta(I;L^p)} \\
& + CT^{1/\rho-1/.\theta}\|f(\tau)\|_{L^\theta(I;L^r)} \\
\leq & M + 9C(\mu_1+1)T^\delta M(M+N+G) + C\mu_1(T^{1/\rho}+T^\delta)M,
\end{aligned}
$$

where

$$
\frac{1}{r} = \frac{2}{p}-\frac{1}{n}, \qquad \frac{1}{\rho}-\frac{1}{\theta} = \frac{1}{2}-\gamma = 1-\frac{n}{2p} \equiv \delta > 0.
$$

Hence again by choosing T sufficiently small, we conclude

$$
\|\nabla^\alpha\Phi(u,v)\|_{L^\theta(I;L^p)} \leq \frac{3}{2}M. \tag{4.6}
$$

Inequalities (4.4) and (4.6) imply (3.2). The inequality (3.3) follows in a simlar way.

5. A priori estimates for the global existence

In this section, we give an apriori estimate for the solution. The space where we have the an apriori estimate is in $L^2(\mathbb{R}^n)$. For simplicity, we let $\mu_1=\mu_2=1$ (n,p)

$v = n + p$, $w = n - p$

$$
(5.1) \qquad \begin{cases} \partial_t v - \nu \Delta n + \nabla(w \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ \partial_t w - \nu \Delta w - \nabla(v \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ -\Delta \psi = -\dfrac{1}{\varepsilon^2}(w + g), \quad x \in \mathbb{R}^n, \\ \qquad v(0,x) = v_0(x) = n_0(x) + p_0(x), \qquad w(0,x) = w_0(x) = n_0(x) - p_0(x). \end{cases}
$$

Multiply the first equatino of (5.1) by v and integral by parts, we have

$$
(5.2) \qquad \frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \int_{\mathbb{R}^n} v \nabla w \cdot \nabla \psi dx + \int_{\mathbb{R}^n} vw \Delta w dx = 0.
$$

By the thrid equation of (5.1),

$$
(5.3) \qquad \frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \int_{\mathbb{R}^n} v \nabla w \cdot \nabla \psi dx + \int_{\mathbb{R}^n} vw(w+g) dx = 0.
$$

Simlarly

$$
(5.4) \qquad \frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \|\nabla w(t)\|_2^2 + \int_{\mathbb{R}^n} w \nabla v \cdot \nabla \psi dx + \int_{\mathbb{R}^n} vw(w+g) dx = 0
$$

Adding (5.3) and (5.4)

$$
(5.5) \qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\{\|v(t)\|_2^2 + \|w(t)\|_2^2\} + \|\nabla v(t)\|_2^2 + \|\nabla w(t)\|_2^2 \\ &\qquad + \int_{\mathbb{R}^n} \nabla vw \cdot \nabla \psi dx + 2\int_{\mathbb{R}^n} v|w|^2 dx + 2\int_{\mathbb{R}^n} gvw dx = 0 \end{aligned}
$$

$$
(5.6) \qquad \frac{1}{2}\frac{d}{dt}\{\|v(t)\|_2^2 + \|w(t)\|_2^2\} + \|\nabla v(t)\|_2^2 + \|\nabla w(t)\|_2^2 + \int_{\mathbb{R}^n} v|w|^2 dx + \int_{\mathbb{R}^n} gvw dx = 0
$$

$$
(5.7) \qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\{\|v(t)\|_2^2 + \|w(t)\|_2^2\} + \|\nabla v(t)\|_2^2 + \|\nabla w(t)\|_2^2 + \int_{\mathbb{R}^n} v|w|^2 dx \\ &\qquad = -\int_{\mathbb{R}^n} gvw dx \le \|g\|_\infty \|v\|\|w\| \le C(\|v\|_2^2 + \|w\|_2^2) \end{aligned}
$$

$$
(5.8) \qquad \frac{d}{dt}\{\|v(t)\|_2^2 + \|w(t)\|_2^2\} \le C(\|v(t)\|_2^2 + \|w(t)\|_2^2)
$$

$$
(5.9) \qquad \|v(t)\|_2^2 + \|w(t)\|_2^2 \le (\|v_0\|_2^2 + \|w_0\|_2^2)e^{ct}
$$

(5.7)

$$
(5.10) \qquad \begin{aligned} &\frac{d}{dt}\{\|v(t)\|_2^2 + \|w(t)\|_2^2\} + 2\|\nabla v(t)\|_2^2 + \|\nabla w(t)\|_2^2 + \int_{\mathbb{R}^n} v|w|^2 dx \\ &\qquad \le C(\|v_0\|_2^2 + \|w_0\|_2^2)e^{Ct} \end{aligned}
$$

Integrate over $[0,t]$ in t variable,

$$\|v(t)\|_2^2+\|w(t)\|_2^2+2\int_0^t\{\|\|\nabla v(t)\|_2^2+\|\nabla w(t)\|_2^2\}d\tau+2\int_0^t d\tau\int_{\mathbb{R}^n} v|w|^2dx \le (\|v_0\|_2^2+\|w_0\|_2^2)e^{Ct} \tag{5.11}$$

It now follows the estimate for $n(t)$ and $p(t)$ that

$$\begin{aligned}&\|n(t)+p(t)\|_2^2+\|n(t)-p(t)\|_2^2+2\int_0^t\{\|\nabla(n(t)+p(t))\|_2^2+\|\nabla(n(t)-p(t))\|_2^2\}d\tau\\&\quad+2\int_0^t d\tau\int_{\mathbb{R}^n}(n(t)+p(t))|n(t)-p(t)|^2dx\le(\|n_0+p_0\|_2^2+\|n_0-p_0\|_2^2)e^{Ct}.\end{aligned} \tag{5.12}$$

Namely

$$\begin{aligned}&\|n(t)\|_2^2+\|p(t)\|_2^2+2\int_0^t\{\|\nabla n(t)\|_2^2+\|\nabla p(t)\|_2^2\}d\tau\\&\quad+\int_0^t d\tau\int_{\mathbb{R}^n}(n(t)+p(t))|n(t)-p(t)|^2dx\le(\|n_0\|_2^2+\|p_0\|_2^2)e^{Ct}.\end{aligned} \tag{5.13}$$

All the above procedure is valid for the sufficiently smooth solution. Now by wellposedness theorem, we have for the a prioiri estimate in L^2.

Proposition 5.1. *Suppose that $g\in L^\infty(\mathbb{R}^n)$ and let $(n_0,p_0)\in L^2\times L^2$. Then the solution obtained in Theorem 1.1 (n,p) satisies the following estimate.*

$$\begin{aligned}&\|n(t)\|_2^2+\|p(t)\|_2^2+2\int_0^t\{\|\nabla n(t)\|_2^2+\|\nabla p(t)\|_2^2\}d\tau\\&\quad+\int_0^t d\tau\int_{\mathbb{R}^n}(n(t)+p(t))|n(t)-p(t)|^2dx\le(\|n_0\|_2^2+\|p_0\|_2^2)e^{Ct}.\end{aligned} \tag{5.14}$$

According to Proposition 5.1, we now obtain the global existence for the positive solution in 3 dimensional case as follows.

Theorem 5.2. *Supporse that $g\in L^\infty(\mathbb{R}^3)$, $f\equiv 0$ and let $(n_0,p_0)\in L^2\times L^2$. Moreover we assume that the initial data are positive definite, i.e, $n_0(x),p_0(x)\ge 0$. Then the solution obtained in Theorem 1.1 (n,p) in $C([0,T);W^{2,2})\cap C^1((0,T);L^2)$ is globally exists.*

Proof of 5.2. By the maximum principle, the local solution obtained in Theorem 1.1 is positive for all time. i.e., $n(t,x)\ge 0$ and $p(t,x)\ge 0$. Then by the a priori estimate Proposition 5.1, we have

$$\|n(t)\|_2^2+\|p(t)\|_2^2+2\int_0^t\{\|\nabla n(t)\|_2^2+\|\nabla p(t)\|_2^2\}d\tau\le(\|n_0\|_2^2+\|p_0\|_2^2)e^{Ct}. \tag{5.15}$$

and this implies the solution can not blow up in a finite time. This concludes the theorem. $\square$

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